

One dimensional random walks killed on a finite set

Kôhei UCHIYAMA

Department of Mathematics, Tokyo Institute of Technology

Oh-okayama, Meguro Tokyo 152-8551

e-mail: uchiyama@math.titech.ac.jp

running head: random walk killed on a finite set

key words: exterior domain; transition probability; escape from a finite set; hitting probability of a finite set; potential theory

AMS Subject classification (2010): Primary 60G50, Secondary 60J45.

Abstract

We study the transition probability, say $p_A^n(x, y)$, of a one-dimensional random walk on the integer lattice killed when entering into a non-empty finite set A . The random walk is assumed to be irreducible and have zero mean and a finite variance σ^2 . We show that $p_A^n(x, y)$ behaves like $[g_A^+(x)\widehat{g}_A^+(y) + g_A^-(x)\widehat{g}_A^-(y)](\sigma^2/2n)p^n(y-x)$ uniformly in the regime characterized by the conditions $|x| \vee |y| = O(\sqrt{n})$ and $|x| \wedge |y| = o(\sqrt{n})$ generally if $xy > 0$ and under a mild additional assumption about the walk if $xy < 0$. Here $p^n(y-x)$ is the transition kernel of the random walk (without killing); g_A^\pm are the Green functions for the ‘exterior’ of A with ‘pole at $\pm\infty$ ’ normalized so that $g_A^\pm(x) \sim 2|x|/\sigma^2$ as $x \rightarrow \pm\infty$; and \widehat{g}_A^\pm are the corresponding Green functions for the time-reversed walk.

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1 Introduction and main results

This paper concerns the transition probability of a one-dimensional random walk on the integer lattice \mathbb{Z} killed on a finite set A . For random walks on the d -dimensional integer lattice \mathbb{Z}^d , $d \geq 1$ killed on a finite set H . Kesten [7] obtained, among others, the asymptotic form of the transition probability under a quite general setting. For the important case of one dimensional random walk with zero mean and finite variance, however, his result is restricted to the special case when A consists of a single point. In this paper we extend it to every finite set A . Our result is stronger than his in another respect: the asymptotic estimate is valid uniformly for space variables within a reasonable range of relevant variables. It is incidentally revealed that according as the third absolute moment of the increment variable is finite or not, the walk killed at the origin exhibits qualitatively different behaviour as the starting and landing positions are taken far from the origin in the opposite directions from each other (see Remark 2 near the end of this section).

Our method of proof is quite different from that of Kesten [7]. In [7] a compactness argument is used as a basic tool. Our proof reflects the behaviour of random walk path. It rests on the results of [12] in which the same problem as the present paper is studied but with $A = \{0\}$ and an asymptotic form of the transition probability valid uniformly for space variables is obtained. The same method is applied in [13] to higher dimensional random walks to obtain a similar strengthening of Kesten's result. For multidimensional Brownian motions the corresponding problem is studied by [3] for space variables restricted to compact sets and by [14] without any restriction as such.

Let $S_n = S_0 + X_1 + \dots + X_n$, $n = 1, 2, \dots$ be a random walk on the one-dimensional integer lattice \mathbb{Z} . Here the increments X_j are i.i.d. \mathbb{Z} -valued random variables defined on some probability space (Ω, \mathcal{F}, P) and the initial state S_0 is an integer left unspecified for now. As usual the law with $S_0 = x$ of the walk (S_n) is denoted by P_x and the corresponding expectation by E_x . Throughout this paper we suppose that the random walk (S_n) is irreducible, namely for every $x \in \mathbb{Z}$, $P_0[S_n = x] > 0$ for some $n > 0$, and that

$$EX = 0 \quad \text{and} \quad 0 < \sigma^2 := EX^2 < \infty \quad (1.1)$$

Here as well as in what follows X is a random variable having the same law as X_1 and E the expectation w.r.t. P .

Let $p(x) = P[X = x]$ and $p^n(x) = P_0[S_n = x]$ so that for $y \in \mathbb{Z}$,

$$P_x[S_n = y] = p^n(y - x) \quad \text{and} \quad p^0(x) = \delta_{x,0},$$

where $\delta_{x,y}$ equals unity if $x = y$ and zero if $x \neq y$. For a non-empty finite subset A of \mathbb{Z} , let $p_A^n(x, y)$ denote the transition probability of the walk S_n killed upon entering A , defined by

$$p_A^n(x, y) = P_x[S_k \notin A \text{ for } 1 \leq k \leq n \text{ and } S_n = y], \quad n = 0, 1, 2, \dots$$

Thus $p_A^0(x, y) = \delta_{x,y}$ (even if $y \in A$); and $p_A^n(x, y) = 0$ whenever $y \in A, n \geq 1$.

Let $a(x)$ be the potential function of the walk defined by

$$a(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n [p^k(0) - p^k(-x)].$$

It is convenient to bring in

$$a^\dagger(x) := \delta_{x,0} + a(x).$$

The result of Kesten [7] mentioned above implies that for each x and $y \neq 0$, as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \frac{p_{\{0\}}^n(x, y)}{f_0(n)} = a^\dagger(x)a^\dagger(-y) + \frac{xy}{\sigma^4}, \quad (1.2)$$

provided that the walk is (temporally) aperiodic in addition. Here

$$f_0(n) = P_0[S_k \neq 0 \text{ for } k = 1, \dots, n-1 \text{ and } S_n = 0].$$

We know that

$$f_0(n) = \frac{\sigma^2}{n} p^n(0) \{1 + o(1)\}$$

as $n \rightarrow \infty$. (Cf. [10] for the existence of $a(x)$ and [7, Theorem 8] for the asymptotic form of $f_0(n)$ stated above.) Note that $y = 0$ is reasonably excluded in (1.2) (cf. Remark 1 (e)).

Denote the Green function of the killed walk by $g_A(x, y)$:

$$g_A(x, y) = \sum_{n=0}^{\infty} p_A^n(x, y), \quad x, y \in \mathbb{Z}.$$

(To be precise this does not conform to the usual nomenclature, according to which a Green function is set zero on $A \times A$, while our $g_A(x, y)$ is equal to $\delta_{x,y}$ if $y \in A$ and to the probability that the first entrance into A takes place at x for the dual walk starting at y if $x \in A, y \notin A$.) According to Theorem 30.1 of [10] $g_A(x, y)$ has limits as $y \rightarrow +\infty$ and $y \rightarrow -\infty$ for each x . We call them

$$g_A^+(x) = \lim_{y \rightarrow \infty} g_A(x, y) \quad \text{and} \quad g_A^-(x) = \lim_{y \rightarrow -\infty} g_A(x, y).$$

($g_A^\pm(x)$ may be interpreted as the expected number of visits to x made by the dual—or time-reversed—random walk ‘started at $\pm\infty$ ’ up to and including the first entrance time into A .) We write $-A$ for $\{-z : z \in A\}$; $s \wedge t$ and $s \vee t$ for the minimum and maximum, respectively, of real numbers s and t . In the following theorem we impose on the pair of x, y the condition

$$g_A^+(x)g_{-A}^-(-y) + g_A^-(x)g_{-A}^+(-y) \neq 0. \quad (1.3)$$

Theorem 1. *Let A be a non-empty finite subset of \mathbb{Z} .*

(i) *For each $M \geq 1$, uniformly for $x \in \mathbb{Z}$ and $y \in \mathbb{Z} \setminus A$ subject to condition (1.3) and the constraints $-M \leq x \leq M\sqrt{n}$ and $-M \leq y \leq M\sqrt{n}$, as $n \rightarrow \infty$ and $(|x| \wedge |y|)/\sqrt{n} \rightarrow 0$*

$$p_A^n(x, y) = \frac{g_A^+(x)g_{-A}^-(-y) + g_A^-(x)g_{-A}^+(-y)}{2n/\sigma^2} p^n(y - x) \{1 + o(1)\}; \quad (1.4)$$

if condition (1.3) is violated, then $p_A^n(x, y) < Ce^{-\lambda n}$ for some positive constants λ and C that depend only on p and A .

(ii) *As $x \wedge y \wedge n \rightarrow \infty$ under $x \vee y < M\sqrt{n}$ along with $p^n(y - x) > 0$*

$$p_A^n(x, y) = \frac{\nu}{\sqrt{2\pi\sigma^2 n}} \left(e^{-(y-x)^2/2\sigma^2 n} - e^{-(y+x)^2/2\sigma^2 n} \right) \{1 + o(1)\}, \quad (1.5)$$

where ν designates the (temporal) period of the random walk.

In the theorem above as well as in what follows, $o(1) \rightarrow 0$ in the specified procedure of taking limit and the convergence is uniform under the specified constraint. By symmetry the results of Theorem 1 are valid if x and y are simultaneously replaced by $-x$ and $-y$, respectively, and this remark applies to the succeeding results. The case $xy < 0$ and $|x| \wedge |y| \rightarrow \infty$, excluded in Theorem 1, is discussed later in this introduction; as a matter of fact formula (1.4) remains true under a mild additional assumption about p but may break down without it (cf. Theorem 2 below and Theorem 3 and Remark 6 in Section 6).

REMARK 1. (a) If the object corresponding to the dual (time-reversed) walk is indicated by putting $\hat{}$, like $\hat{p}_A^n(y, x)$, we have for $x, y \notin A$

$$g_{-A}(-y, -x) = \hat{g}_A(y, x) = g_A(x, y),$$

hence

$$g_{-A}^-(y) = \hat{g}_A^+(y) = \lim_{x \rightarrow \infty} g_A(x, y) \quad (1.6)$$

(cf. [10, Section 10]; see also Appendix A for explanation and related matters).

(b) We shall show

$$g_A^\pm(x) = a(x) \pm x/\sigma^2 + O(1),$$

(see (3.3), (3.4)). Here as elsewhere both upper or both lower signs should be chosen in the double signs. It in particular follows that

$$g_A^+(x)/x \longrightarrow 2/\sigma^2 \text{ or } 0 \quad \text{according as } x \rightarrow +\infty \text{ or } -\infty;$$

and similarly for $g_A^-(x)$ and $\hat{g}_A^\pm(y)$.

By substitution of these relations formula (1.4) is somewhat simplified when $x \vee y \rightarrow \infty$ (under $x \vee y < M\sqrt{n}$ and $x \wedge y = o(\sqrt{n})$). In fact, in the case $y \rightarrow \infty$, we have $g_{-A}^+(-y) = o(y)$ as well as $g_{-A}^-(-y) \sim y/\sigma^2$ and, if (1.3) holds, $p_A^k(x, y) > 0$ for some $k \geq 1$, which implies $g_A^+(x) > 0$ (cf. (3.8)), so that (1.4) can be written as

$$p_A^n(x, y) \sim \frac{g_A^+(x)y}{n} p^n(y - x), \quad (1.7)$$

where the symbol \sim means that the ratio of two sides of it approaches unity; and similarly for the case $x \rightarrow \infty$ (cf. Lemma 5.2).

Also, as $x \wedge y \rightarrow \infty$ (under $x \vee y < M\sqrt{n}$ and $x \wedge y = o(\sqrt{n})$), (1.4) reduces to

$$p_A^n(x, y) \sim \frac{2xy}{\sigma^2 n} p^n(y - x).$$

This relation conforms to (1.4) in view of a local central limit theorem. It also follows that the restriction $x \wedge y = o(\sqrt{n})$ cannot be relaxed in (1.4). Generally in the parabolic regime $|x| \vee |y| = O(\sqrt{n})$ it holds that

$$p_A(x, y) \asymp |xy|n^{-1}p^n(y - x) \quad \text{if } xy > 0$$

(\asymp means that the ratio of two sides is bounded away from zero and infinity) and

$$C(|x| + |y|)n^{-1}p^n(y - x) \leq p_A(x, y) = p^n(y - x) \times o(|xy|/n) \quad \text{if } xy < 0,$$

provided condition (1.3) is satisfied. (See Theorem 2 and Remark 2 after it for the latter.)

(c) Because of (1.6) $g_{-A}^{-}(-y)$ (resp. $g_{-A}^{+}(-y)$) is positive if and only if y can be reached by the walk starting at $+\infty$ (resp. $-\infty$). Taking account of this and its analogue for g_A^{\pm} we see that the product $g_A^{+}(x)g_{-A}^{-}(-y)$ (resp. $g_A^{-}(x)g_{-A}^{+}(-y)$) is positive if (and only if) the walk starting at x can reach y after a large excursion to the right (resp. left) with a positive probability. Hence, if both of these two products vanish, namely if condition (1.3) is violated, then one of the following (1) or (2) must hold true:

(c1) $p_A^n(x, y)$ vanish for all n ;

(c2) all the random walk paths up to σ_A that start at x and pass through y are confined in the convex hull of A with probability one,

where σ_A denotes the first entrance time into A (see (1.9) below for the precise definition). In the second case the paths must enter A in a small number of steps so as to ensure the second assertion of Theorem 1.

(d) Formula (1.4) may be equivalently stated as follows: The probability that the walk (S_n) started at x and pinned at y at time n avoids A is asymptotically equivalent to the ratio on the right side of (1.4), namely

$$P_x[\sigma_A > n \mid S_n = y] \sim \frac{g_A^{+}(x)g_{-A}^{-}(-y) + g_A^{-}(x)g_{-A}^{+}(-y)}{2n/\sigma^2}$$

for n, x, y such that $p^n(y - x) > 0$.

(e) Formula (1.4) implies Kesten's result (1.2). Indeed, $g_{\{0\}}^{\pm}(x) = a^{\dagger}(x) \pm x/\sigma^2$ so that for $A = \{0\}$ the numerator of the ratio on the right side of (1.4) reduces to $a^{\dagger}(x)a^{\dagger}(-y) + xy/\sigma^4$ (cf. (1.11), (3.3)). What is actually treated, instead of our $p_A^n(x, y)$, in [7] is

$$Q_A^n(x, y) := P_x[S_k \notin A \text{ for } 0 < k < n \text{ and } S_n = y], \quad n = 0, 1, 2, \dots,$$

so that $Q_A^n(x, y)$ equals $p_A^n(x, y)$ or $P_x[\sigma_A = n, S_{\sigma(A)} = y]$ according as $y \notin A$ or $y \in A$. If $p_{\{0\}}^n(x, y)$ is replaced by $Q_{\{0\}}^n(x, y)$, then formula (1.2) becomes valid also for $y = 0$. The same remark applies to (1.4) for $y \in A$ as formulated below in Corollary 1. The corresponding Green function $G_A(x, y) := \sum_{n=0}^{\infty} Q_A^n(x, y)$ is related to g_A by

$$G_A(x, y) = g_A(x, y) + P_x[S_{\sigma(A)} = y]. \quad (1.8)$$

Although Q_A^n and G_A are natural objects to consider because of their symmetry relative to duality, we adhere to $p_A^n(x, y)$ as the principal object to study.

(f) For random walks with drift (i.e. in case $\mu := EX \neq 0$) one can readily derive from Theorem 1 the corresponding asymptotic formulae (namely those in the regime $|x| \vee |y| = O(\sqrt{n})$ if $\sum p(x)s_0^x x = 0$ and $\sum p(x)s_0^x x^2 < \infty$ for some $s_0 \neq 1$. In the case when there exists no such s_0 Kesten's results [7, Theorems 5 and 6] provide a certain asymptotic formula under an additional condition. From another view point it is natural to take up the regime $|y - x - \mu n| \leq M\sqrt{n}$. Without difficulty one can show that if $\mu > 0$, then for each $\delta > 0$, uniformly for $x \geq (-\mu + \delta)n$ as $n \rightarrow \infty$ and y in this regime

$$p_A^n(x, y) \sim P_x[\sigma_A = \infty]p^n(y - x).$$

Some detailed investigation is undertaken in a separate paper [15].

(g) Extensions to non-lattice walks would be interesting. The present work rests on the results for the case $A = \{0\}$ given in [12] for which the harmonic analysis is effectively

applicable but does not seem to be for the non-lattice walks. However, another approach seems promising to lead to corresponding results at least under some assumption on the distribution of the increment variable: one may take the half line $(-\infty, 0]$ in place of $\{0\}$ (see Remark 4 of Section 5) and apply the corresponding results for the transition probability that are found in several papers [1], [2], [4], [16], etc. and those for the potential operator given in [9].

For a non-empty set $B \subset \mathbb{Z}^d$, σ_B (resp. τ_B) denotes the first time when S_n enters into (resp. exits from) B :

$$\sigma_B = \inf\{n \geq 1 : S_n \in B\}, \quad \tau_B = \inf\{n \geq 1 : S_n \notin B\}. \quad (1.9)$$

For typographical reason we shall sometimes write $\sigma(B)$ for σ_B and similarly for $\tau(B)$. For a positive integer R denote the interval $\{-R+1, \dots, R-2, R-1\}$ by

$$U(R).$$

The roles of g_A^\pm and g_{-A}^\pm appearing in (1.4) may be explained by the formula

$$P_x[\tau_{U(R)} = \sigma_{[R,\infty)} < \sigma_A] = \frac{g_A^+(x)}{R} \{1 + o(1)\} \quad (1.10)$$

(Proposition 4 (§2)) and its reverse and dual formulae (see the paragraph given near the end of this section for a little more details). Here $o(1) \rightarrow 0$ as $R \rightarrow \infty$ uniformly for $-M < x \leq R$. Indeed formula (1.10) is a gist of the proof of Theorem 1, and Section 3 will be devoted to its proof.

For $\xi_0 \in A$ and $x \in \mathbb{Z}$, put

$$w_A(x) = \sigma^{-2}(x - E_x[S_{\sigma(A)}])$$

and

$$u_A(x) = a^\dagger(x - \xi_0) - E_x[a(S_{\sigma(A)} - \xi_0)],$$

where the right side does not depend on the choice of ξ_0 (cf. [13], see also (3.1)). We shall see that

$$g_A^+(x) = u_A(x) + w_A(x) \quad \text{and} \quad g_A^-(x) = u_A(x) - w_A(x), \quad (1.11)$$

by which we obtain the identity

$$\frac{g_A^+(x)g_{-A}^-(-y) + g_A^-(x)g_{-A}^+(-y)}{2} = u_A(x)u_{-A}(-y) - w_A(x)w_{-A}(-y).$$

By substitution formula (1.4) becomes quite analogous to the one for the case $A = \{0\}$ as given in Theorem A (i) of Section 4.1 that extends (1.2) to unbounded x, y .

Noting that $g_A(\cdot, y)$ is bounded for each y , we pass to the limit in the identity

$$\sum_{z \in \mathbb{Z}} p_A^1(x, z) g_A(z, y) = g_A(x, y) - \delta_{x, y}$$

to find that g_A^+ is a harmonic function for the killed walk in the sense that

$$g_A^\pm(x) = \sum_{z \in \mathbb{Z} \setminus A} p(z - x) g_A^\pm(z) \quad \text{for all } x \in \mathbb{Z}; \quad (1.12)$$

the functions u_A and w_A also are harmonic in the same sense, and $g_A^\pm(-\cdot)$, $u_{-A}(-\cdot)$ and $w_{-A}(-\cdot)$ are dual harmonic.

From the proof of Theorem 1 or directly by a usual argument based on the last leaving decomposition with the help of the relations dual to (1.12) (see Remark 5 at the end of Section 5) we obtain the following

Corollary 1. *Let A be a non-empty finite subset of \mathbb{Z} . Then, for each $M > 1$ and for $\xi \in A$, as $n \rightarrow \infty$*

$$P_x[\sigma_A = n, S_{\sigma(A)} = \xi] = \frac{g_A^+(x)g_{-A}^-(-\xi) + g_A^-(x)g_{-A}^+(-\xi)}{2n/\sigma^2} p^n(\xi - x) \{1 + o(1)\} \quad (1.13)$$

uniformly for $x \in \mathbb{Z}$ satisfying $|x| < M\sqrt{n}$ and $g_A^+(x)g_{-A}^-(-\xi) + g_A^-(x)g_{-A}^+(-\xi) > 0$; if the latter condition is violated, then the probability on the left side of (1.13) either vanishes for every n or tends to zero exponentially fast as $n \rightarrow \infty$.

For a non-empty set B that is contained in $(-\infty, N]$ for some N , we put

$$H_B^{+\infty}(y) = \lim_{x \rightarrow \infty} P_x[S_{\sigma(B)} = y]$$

and similarly for $H_B^{-\infty}$ if $B \subset [N, \infty)$ (the limits exist and $H_B^{\pm\infty}$ are probabilities as is established in [10, Theorem 30.1]). It is noted that $g_A^+(\xi) = \sum_{z \notin A} p(z - \xi)g_A^+(z)$ is the probability that the dual walk ‘started at $+\infty$ ’ hits A at ξ , which fact is expressed as

$$g_A^+(\xi) = H_{-A}^{-\infty}(-\xi), \quad \xi \in A,$$

and similarly $g_A^-(\xi) = H_{-A}^{+\infty}(-\xi)$, representing the same probability but for the dual walk started at $-\infty$. Thus $g_{-A}^-(-\xi)$ is the probability that the dual walk ‘started at $-\infty$ ’ hits $-A$ at $-\xi$, whence

$$g_{-A}^-(-\xi) = H_A^{+\infty}(\xi), \quad \xi \in A, \quad (1.14)$$

and similarly $g_{-A}^+(-\xi) = H_A^{-\infty}(\xi)$. By (1.14) and by (1.11) we especially have

$$\sum_{\xi \in A} g_A^+(\xi) = \sum_{\xi \in A} g_A^-(\xi) = 1.$$

It is noted that

$$\frac{g_A^+(x) + g_A^-(x)}{2} = u_A(x).$$

The asymptotic form of $P_x[\sigma_A = n]$ is obtained by making summation over $\xi \in A$ in (1.13). In general, however, we need to take care of the temporal periodicity of the walk and partition the lattice \mathbb{Z} according to it. In the following corollary we suppose for simplicity that the random walk is aperiodic (strongly aperiodic in the sense of [10]) so that $p^n(\xi - x)$ may be replaced by $p^n(-x)$.

Corollary 2. *Suppose the random walk is aperiodic. Then*

$$P_x[\sigma_A = n] = \frac{\sigma^2 u_A(x)}{n} p^n(-x) \{1 + o(1)\}$$

as $n \rightarrow \infty$; and for $\xi \in A$,

$$\lim_{n \rightarrow \infty} P_x[S_{\sigma(A)} = \xi \mid \sigma_A = n] = \frac{g_A^+(x)}{g_A^+(x) + g_A^-(x)} H_A^{+\infty}(\xi) + \frac{g_A^-(x)}{g_A^+(x) + g_A^-(x)} H_A^{-\infty}(\xi).$$

Here $o(1)$ and the convergence in the $\lim_{n \rightarrow \infty}$ are uniform for $|x| < M\sqrt{n}$.

It is often useful to have an upper bound of $p_A^n(x, y)$ valid for all x, y . The following one is essentially a corollary of [12, Theorem 1.1] (see the end of Section 4.1 for the proof).

Proposition 1. *There exists a constant C such that if $|x| \wedge |y| \wedge n \geq 1$,*

$$p_A^n(x, y) \leq \frac{|x| \vee |y|}{|x| \vee |y| \vee \sqrt{n}} \left[C \frac{(|x| \wedge |y|) \mathbf{g}_{4n}(y-x)}{(|x| \vee |y| \vee \sqrt{n})} + o\left(\frac{|x| \wedge |y| \wedge \sqrt{n}}{(y-x)^2 \vee n}\right) \right], \quad (1.15)$$

where $\mathbf{g}_n(t) = (2\pi\sigma^2 n)^{-1/2} e^{-t^2/2\sigma^2 n}$.

The case when $y < 0 < x$ and $x \wedge |y| \rightarrow \infty$ —excluded from Theorem 1—requires somewhat delicate analysis to find whether formula (1.4) holds true. We shall prove in Theorem 3 that (1.4) remains valid for $x > 0, y < 0$ satisfying the constraints $|x| \vee |y| < M\sqrt{n}$ and $|x| \wedge |y| = o(\sqrt{n})$ under a mild additional condition on p which is true at least if the tail $F(y) = P[X < y]$ is regularly varying at $-\infty$ with an exponent less than -2 . For its proof we need to know a certain property of the difference $a(z) - z/\sigma^2$ (not $a(z) + z/\sigma^2$ as in the case $y > 0$) for large positive values of z , which is sensitive to the behaviour of the tail $P[X < -x]$ for large x in case $E[|X|^3; X < 0] = \infty$ (cf. Lemma 2.1). Under the assumption $E[|X|^3; X < 0] < \infty$, however, things are particularly simplified as given below.

Put

$$C^+ := \sum_{y=-\infty}^0 H_{(-\infty, 0]}^{+\infty}(y)(\sigma^2 a(y) + |y|) \leq \infty.$$

It holds that

$$C^+ = \lim_{x \rightarrow +\infty} (\sigma^2 a(x) - x), \quad (1.16)$$

$C^+ < \infty$ if and only if $E[|X|^3; X < 0] < \infty$, and $C^+ > 0$ unless the walk is *left-continuous*, i.e., unless $P[X \leq -2] = 0$ (cf. Corollary 2.1 and Remark 1 (a) of [12]). We call

$$C_A^+ := \sigma^2 \lim_{x \rightarrow +\infty} g_A^-(x) = C^+ - \sum_{\xi \in A} H_A^{+\infty}(\xi) [\sigma^2 a(\xi - \xi_0) - (\xi - \xi_0)]. \quad (1.17)$$

(Of course $\xi_0 \in A$; the sum does not depend on the choice of ξ_0 (see (6.1)).) It will be proved that $C_{-A}^+ = C_A^+$, namely

$$C_A^+ = \sigma^2 \lim_{x \rightarrow +\infty} g_{-A}^-(x) \quad (1.18)$$

(Lemma 7.1) and that $C_A^+ > 0$ unless $g_A^-(x)$ vanishes for all sufficiently large (positive) x (Lemma 3.2). By (1.11) $\sigma^2 g_A^+(x) \sim 2x$ ($x \rightarrow \infty$) and $\sigma^2 g_{-A}^+(-y) \sim 2|y|$ ($y \rightarrow -\infty$). Putting these together yields that if $C_A^+ < \infty$, then as $x \wedge (-y) \rightarrow \infty$

$$\frac{g_A^+(x)g_{-A}^+(-y) + g_A^-(x)g_{-A}^+(-y)}{2} = C_A^+ \frac{x + |y|}{\sigma^4} \{1 + o(1)\}.$$

In view of this relation the next result is a natural extension of Theorem 1 to the case $(-y) \wedge x \rightarrow \infty$; it also extends Theorem 1.2 of [12] where the same result is obtained for $A = \{0\}$.

Theorem 2. *If $E[|X|^3; X < 0] < \infty$, then as $x \wedge (-y) \wedge n \rightarrow \infty$ subject to the condition $x \vee (-y) < M\sqrt{n}$*

$$p_A^n(x, y) = C_A^+ \frac{x + |y|}{\sigma^2 n} p^n(y - x) \{1 + o(1)\}.$$

REMARK 2. (a) In Section 6 we shall see the following results: within the parabolic regime $-M\sqrt{n} < y < 0 < x < M\sqrt{n}$

$$(a1) \quad p_A^n(x, y) \leq C\{g_A^+(x)g_{-A}^-(y) + g_{-A}^-(x)g_A^+(y)\}n^{-3/2}$$

$$(a2) \quad p_{\{0\}}^n(x, y) \geq C\{(x + |y|)n^{-3/2}\} \sum_{w=2}^{x \wedge |y|} p(-w)w^3 \quad (x \wedge |y| \geq 2)$$

(according to Propositions 7 and 8, respectively).

(b) Suppose that $0 \in A$ and $\sharp A \geq 2$ ($\sharp A$ denotes the cardinality of A) and ask in what situations $p_A^n(x, y)$ exhibits different asymptotic behaviour than $p_{\{0\}}^n(x, y)$ does. Suppose the walk is aperiodic for simplicity and restrict the variables to the regime $|x| \vee |y| = O(\sqrt{n})$.

First note that $p_A^n(x, y) = p_{\{0\}}^n(x, y)$ ($y \neq 0$) for all n if and only if every path of the walk from x to $y \neq 0$ that avoids 0 also avoids A ($y = 0$ is reasonably excluded). Preclude from our consideration this trivial situation and the case when $p_{\{0\}}^n(x, y) = 0$ for all $n \geq 1$. Then, as $|x| \vee |y| \vee n \rightarrow \infty$ under the condition $|x| \wedge |y| = O(1)$ the ratio $p_A^n(x, y)/p_{\{0\}}^n(x, y)$ converges to a function of x or y or of both that is less than 1, as one may expect and deduces from Theorem 1 (see Appendix C for a proof and more details). On the other hand if $xy > 0$ and $|x| \wedge |y| \wedge n \rightarrow \infty$, $p_A^n(x, y) \sim p_{\{0\}}^n(x, y)$, so that the asymptotic form of $p_A^n(x, y)$ does not depend on A in this regime as is obvious from Theorem 1. This means that the difference between these two probabilities is negligible when compared with each of them, or what is the same thing, the conditional probability

$$P_x[\sigma_A > n \mid \sigma_{\{0\}} > n, S_n = y] \quad (1.19)$$

approaches unity for every finite set A .

The result in case $xy < 0$ and $|x| \wedge |y| \rightarrow \infty$ may be in a sense more interesting and worthy of note. Proposition 5 in Section 6 entails that under $x \vee |y| \leq M\sqrt{n}$,

$$p_{\{0\}}^n(x, y) - p_A^n(x, y) = O((|x| \vee |y|)n^{-3/2}) \quad \text{if } xy < 0$$

Combined with Theorem 2 and (a2) above this shows that as $x \wedge (-y) \rightarrow \infty$

$$\frac{p_A^n(x, y)}{p_{\{0\}}^n(x, y)} \longrightarrow \begin{cases} C_A^+/C^+ & \text{if } E[|X|^3; X < 0] < \infty, \\ 1 & \text{if } E[|X|^3; X < 0] = \infty, \end{cases} \quad (1.20)$$

thus the asymptotic form of $p_A^n(x, y)$ in this regime does or does not depend on A according as $|x|^3 p(x)$ is summable on $x < 0$ or not, a consequence suggested by the result in [12] mentioned just before Theorem 2 but not fully expected to be true. One may wonder how the difference exhibited in (1.20) arises. This time the origin lies between x and y so that the walk must jump over the origin to avoid it whether the jump is short or long. Now the difference in question is understood to accord with different behaviour of the walk: in the first case the walk that reaches y gets close to the origin, yet avoids it, whereas in the second case it clears the origin by a very long jump so as to avoid A simultaneously (see Corollary 7 for more definite formulation).

We conclude this section by describing the main steps of derivation of the formula of Theorem 1 (i) restricted to the case $|x| \vee |y| = o(\sqrt{n})$. As mentioned before, the proof rests on formula (1.10) and its counter-relation

$$P_x[\tau_{U(R)} = \sigma_{(-\infty, -R]} < \sigma_A] = R^{-1}g_A^-(x)(1 + o(1)).$$

In [12] we have shown that (1.2) holds uniformly in the regime $|x| \vee |y| = o(\sqrt{n})$, which in particular entails that as $|x| \wedge |y| \wedge n \rightarrow \infty$ under the condition $|x| \vee |y| = o(\sqrt{n})$

$$p_{\{0\}}^n(x, y) = \frac{2xy}{\sigma^2 n} p^n(y - x) + o\left(\frac{|x| \vee |y|}{n^{3/2}}\right) \quad \text{if } xy > 0$$

and

$$p_{\{0\}}^n(x, y) = o\left(\frac{|x| \vee |y|}{n^{3/2}}\right) \quad \text{if } xy < 0;$$

moreover, we shall see $p_A^n(x, y) \sim p_{\{0\}}^n(x, y)$ in the same limit scheme. Taking $R = R_n = o(\sqrt{n})$, so that $\tau_{U(R)} = o(n)$ (a.s.), we apply the strong Markov property at $\tau_{U(R)}$ and put all the relations mentioned above together to deduce that for $|x| < |y| = o(\sqrt{n})$, as $|y| \wedge n \rightarrow \infty$

$$p_A^n(x, y) = \frac{g_A^{\text{sign}(y)}(x)|y|}{n} p^n(y - x) + o\left(\frac{(|x| + 1) \vee |y|}{n^{3/2}}\right),$$

where $\text{sign}(y)$ is $+$ or $-$ according as y is positive or negative. Finally on applying the last relation to the time-reversed walk the same argument leading to it shows the formula of the theorem for x and y fixed.

In Section 2 we provide certain potential theoretic facts that are used throughout the paper. In Section 3.1 various relations between g_A , u_A and g_A^\pm are obtained, especially (1.11) is proved. In Section 3.2 we evaluate the probability of the walk exiting $U(R)$ without hitting A for large R and thereby prove (1.10). In Section 4 we state a few known facts on $p_{\{0\}}^n(x, y)$ and $p_{(-\infty, 0]}^n(x, y)$ and prove some results concerning them. Proof of Theorem 1 is given in Section 5. In Section 6 the estimation of $p_A^n(x, y)$ is carried out in case $xy < 0$ and $|x| \wedge |y| \rightarrow \infty$, and in particular the upper and lower bounds in Remark 2 (a) as well as Theorem 2 are proved. We provide in Section 7 some auxiliary results that are used in Remark 2 and Sections 3, 5 and 6.

2 Preliminaries from the theory of one-dimensional random walks

In this section we review some potential theoretic results concerning one-dimensional random walks on \mathbb{Z} that are relevant to the present problem. The most of them are taken from Spitzer's book [10]. Some of the results and the arguments taken from [12] are included to make the description streamlined. We shall designate by C, C', C_1, \dots etc. constants depending only on p whose exact values are not significant for the present purpose and may vary at different occurrences of them. The letters x, y and z are used to denote the integers representing states of the walk.

Potential function. It is shown in [10, Theorem 29.2] that $a(x + 1) - a(x) \rightarrow \pm 1/\sigma^2$ as $x \rightarrow \pm\infty$, which implies

$$a(x + z) - a(x) = \pm \frac{z}{\sigma^2} \{1 + o(1)\}$$

with $o(1) \rightarrow 0$ as $x \rightarrow \pm\infty$ uniformly for z with $(x + z)x > 0$, and $a(x)/|x| \rightarrow 1/\sigma^2$ as $|x| \rightarrow \infty$. It also holds that for all $x, y \in \mathbb{Z}$,

$$\sigma^2 a(x) \geq |x|$$

and

$$\sum_{z \in \mathbb{Z}} p(z-x)a(z-y) = a^\dagger(x-y),$$

that $a(x)$ is linear for $x \geq 0$ (resp. $x \leq 0$) if and only if the walk is left-continuous (resp. right-continuous, i.e., $P[X \geq 2] = 0$) and that

$$\begin{cases} \sigma^2 a(x) > x & \text{for all } x > 0 & \text{if not left-continuous,} \\ \sigma^2 a(-x) > -x & \text{for all } x < 0 & \text{if not right-continuous.} \end{cases} \quad (2.1)$$

(Cf. Theorem 28.1, Proposition 31.1 and Proposition 30.3 of [10] except for the strict inequality (2.1) which is found in [12]: see also (2.3), (2.15) below). In what follows these relations will be used frequently and not noticed of their use.

The results given in the rest of this section will be used in the proofs of Theorem 3 given in Section 6 but not needed for Theorem 1 at least explicitly—some may be used in the proof of Theorem A cited in Section 4.

Green's function on $[1, \infty)$. Here we consider the walk killed when it enters $(-\infty, 0]$. For $x = 1, 2, \dots$, let $v_+(x)$ (resp. $v_-(x)$) be the probability that the strictly ascending (resp. descending) ladder process starting at $+1$ (resp. -1) visits x (resp. $-x$): for $x \geq 2$

$$v_-(x) = P_{-1}[S_{\sigma_{(-\infty, -x]}} = -x] \quad \text{and} \quad v_+(x) = P_1[S_{\sigma_{[x, \infty)}} = x],$$

and $v_+(1) = v_-(1) = 1$, and define

$$f^+(x) = E_0[|S_{\sigma_{(-\infty, -1]}}|](v_-(1) + \dots + v_-(x))$$

and

$$f^-(x) = E_0[S_{\sigma_{[1, +\infty)}}](v_+(1) + \dots + v_+(x)).$$

Then by the renewal theorem $\lim_{x \rightarrow \infty} v_-(x) = 1/E_0[|S_{\sigma_{(-\infty, -1]}}|]$ and similarly for $v_+(x)$, so that $f^+(x) \sim f^-(x) \sim x$ as $x \rightarrow \infty$. Under this boundary condition at $+\infty$, $f^+(x)$ (resp. $f^-(x)$) is the unique harmonic function with respect to the walk S_n (resp. $-S_n$) killed on $(-\infty, 0]$:

$$f^\pm(x) = E[f^\pm(x \pm X); x \pm X > 0] \quad (x \geq 1) \quad \text{and} \quad \lim_{x \rightarrow \infty} f^\pm(x)/x = 1, \quad (2.2)$$

(cf. Sections 18 and 19 of [10]; in particular Proposition 19.5 for the uniqueness). (It is warned that it is not $[1, \infty)$ but $[0, \infty)$ on which the harmonic function is considered in [10].) It also holds [12, Proposition 7.3] that

$$x < f^+(x) < \sigma^2 a(x) \quad \text{for } x \geq 1 \quad (2.3)$$

unless the walk is left continuous (when $f^+(x) = \sigma^2 a(x) = x$ for $x > 0$). The Green function $g_{(-\infty, 0]}$ is expressed as

$$g_{(-\infty, 0]}(x, y) = \frac{2}{\sigma^2} \sum_{z=0}^{x \wedge y - 1} v_+(x-z)v_-(y-z) \quad (x, y > 0). \quad (2.4)$$

Hitting distribution of $(-\infty, 0]$. Let $H_{(-\infty, 0]}^x(y)$ ($y \leq 0$) denote the hitting distribution of $(-\infty, 0]$ for the walk S_n started at x . By the last exit decomposition

$$H_{(-\infty, 0]}^x(y) = \sum_{w=1}^{\infty} g_{(-\infty, 0]}(x, w)p(y-w). \quad (2.5)$$

As mentioned in Introduction $H_{(-\infty,0]}^{+\infty}(y) = \lim_{x \rightarrow \infty} H_{(-\infty,0]}^x(y)$ is a probability. By (2.4)

$$H_{(-\infty,0]}^{+\infty}(y) = \frac{2}{\sigma^2} E[f^-(y-X); X < y] = \frac{2}{\sigma^2} \sum_{w=1}^{\infty} f^-(w)p(y-w) \quad (y \leq 0). \quad (2.6)$$

By summation by parts we have

$$H_{(-\infty,0]}^{+\infty}(y) = \frac{2}{\sigma^2} \sum_{w=1}^{\infty} v_-(w)F(y-w), \quad (2.7)$$

where F denotes the distribution function of X :

$$F(t) = P[X \leq t].$$

From (2.7) it follows that $H_{(-\infty,0]}^{+\infty}(y)$ is monotone and

$$H_{(-\infty,0]}^{+\infty}(y) \asymp \sum_{z < y} F(z);$$

in particular

$$\sum H_{(-\infty,0]}^{+\infty}(y)|y| < \infty \quad \text{if and only if} \quad E[|X|^3; X < 0] < \infty. \quad (2.8)$$

In view of (2.4) $g_{(-\infty,0]}(x, y) \leq Cx \wedge y$ ($x, y > 0$), hence by $f^-(w) \leq Cw$ and (2.6)

$$H_{(-\infty,0]}^x(y) \leq CH_{(-\infty,0]}^{+\infty}(y) \quad (y \leq 0); \quad (2.9)$$

also

$$H_{(-\infty,0]}^x(y) \leq Cx \sum_{w>0} p(y-w) \leq CxF(z) \quad (y < 0). \quad (2.10)$$

Green's function on $\mathbb{Z} \setminus \{0\}$. This is given by

$$\begin{aligned} g(x, y) &:= g_{\{0\}}(x, y) - \delta_{x,0} \\ &= a(x) + a(-y) - a(x-y) \end{aligned} \quad (2.11)$$

([10, Proposition 29.4]). Since $g(x, y) \leq g(x, x) \wedge g(y, y)$,

$$g(x, y) \leq C|x| \wedge |y| \quad \text{for all } x \text{ and } y. \quad (2.12)$$

By the strong Markov property of the walk S we have

$$\sum_{z=-\infty}^0 H_{(-\infty,0]}^x(z)g(z, y) = g(x, y) \quad \text{for } y \leq 0 < x. \quad (2.13)$$

By noting $g(\cdot, y)$ is bounded for each fixed y we let $x \rightarrow \infty$ to obtain

$$\sum_{z=-\infty}^0 H_{(-\infty,0]}^{+\infty}(z)g(z, y) = a(-y) + \frac{y}{\sigma^2}, \quad (2.14)$$

while on letting $y \rightarrow -\infty$ in (2.13) with the help of (2.10) and (2.12)

$$\sum_{z=-\infty}^0 H_{(-\infty,0]}^x(z) \left(a(z) - \frac{z}{\sigma^2} \right) = a(x) - \frac{x}{\sigma^2} \quad \text{for } x > 0. \quad (2.15)$$

In view of (2.9) and (2.8) this shows (1.16) as well as the result stated following it, in particular $\lim_{x \rightarrow \infty} [a(x) - x/\sigma^2]$ is finite if $E[|X|^3; X < 0] < \infty$.

Lemma 2.1. *Suppose $E[|X|^3; X < 0] = \infty$. Then, as $x \rightarrow \infty$*

$$a(x) - \frac{x}{\sigma^2} \sim \frac{2}{\sigma^2} \sum_{z=-x}^{-1} \sum_{w=-\infty}^z H_{(-\infty, 0]}^{+\infty}(w) \quad (2.16)$$

$$\sim \frac{4}{\sigma^4} \sum_{z=-x-1}^{-1} \sum_{w=-\infty}^z \sum_{j=-\infty}^w F(j). \quad (2.17)$$

Proof. Let $y := -x \rightarrow -\infty$ in (2.14). Noting

$$g(z, -x) = \begin{cases} [a(z) - \sigma^{-2}z]\{1 + o(1)\} \sim 2\sigma^{-2}|z| & (-x < z < 0), \\ g(-x, -x)\{1 + o(1)\} \sim 2\sigma^{-2}x & (z \leq -x), \end{cases}$$

where $o(1)$ is uniform in z , we then deduce that if $E[|X|^3; X < 0] = \infty$,

$$a(x) - \frac{x}{\sigma^2} \sim \frac{2}{\sigma^2} \sum_{z=-\infty}^{-1} (|z| \wedge x) H_{(-\infty, 0]}^{+\infty}(z). \quad (2.18)$$

By summation by parts the right side equals that of (2.16) and (2.17) follows immediately from (2.7) and (2.16). The proof is complete. \square

3 The Green function and escape from A

3.1 Functions u_A , g_A^+ and g_A^-

In analogy of the corresponding formula for two-dimensional Brownian motion (cf. [5]) we define $u_A(x)$, $x \in \mathbb{Z}$ by

$$u_A(x) = g_A(x, y) + a(x - y) - E_x[a(S_{\sigma(A)} - y)]. \quad (3.1)$$

This conforms to the definition in Section 1 in view of the next result proved in [13, Lemma 2.8].

Lemma 3.1. *The right side of (3.1) is independent of $y \in \mathbb{Z}^2$ for all $x \in \mathbb{Z}^2$. In particular for any $\xi_0 \in A$*

$$u_A(x) = a^\dagger(x - \xi_0) - E_x[a(S_{\sigma(A)} - \xi_0)]. \quad (3.2)$$

REMARK 3. In [13] it is shown for a two-dimensional walk that the right side of (3.2) is dual-harmonic (i.e. invariant under the transform $h(y) \mapsto \sum_z p(y - z)h(z)$) as a function of $y \in \mathbb{Z}$, hence does not depend on y due to the fact that the only bounded harmonic functions are constant functions [10, Theorem 24.1], [6, Proposition 5-20]. The proof applies without any change to every irreducible recurrent random walk (aperiodic or not) of any dimension. A non-lattice analogue of (3.1) is obtained in [9].

For x restricted on A , (3.1) reduces to the dual of the formula of Proposition 30.1 in [10], where the dual of $u_A(x)$ (denoted by $\mu_A(x)$ therein) is defined as the limit of $\sum_{z \in \mathbb{Z}^2} p^n(z - y)P_z[S_{\sigma(A)} = x]$ as $n \rightarrow \infty$. For $x \notin A$, on the other hand, an equivalent to (3.1) (in a sense) is found in [8, Proposition 4.6.3] where A may be an infinite set while p is assumed to be of

finite range and symmetric. It is warned that our $g_A(x, y)$ is different from Green's function defined in [10] and [8] (see also (1.8)).

Passing to the limit in (3.1) we obtain

$$\begin{aligned} u_A(x) &= \lim_{y \rightarrow +\infty} g_A(x, y) - (x - E_x[S_{\sigma(A)}])/\sigma^2 \\ &= g_A^+(x) - w_A(x), \end{aligned}$$

and similarly $u_A(x) = g_A^-(x) + w_A(x)$, showing (1.11). Substitution from (3.2) then yields expressions of g_A^+ and g_A^- , which we write down as

$$g_A^+(x) = a^\dagger(x - \xi_0) + \frac{x}{\sigma^2} - E_x \left[a(S_{\sigma(A)} - \xi_0) + \frac{1}{\sigma^2} S_{\sigma(A)} \right] \quad (3.3)$$

and

$$g_A^-(x) = a^\dagger(x - \xi_0) - \frac{x}{\sigma^2} - E_x \left[a(S_{\sigma(A)} - \xi_0) - \frac{1}{\sigma^2} S_{\sigma(A)} \right], \quad (3.4)$$

where ξ_0 is a point arbitrarily chosen from A .

As $y \rightarrow \infty$ we have $a(S_{\sigma(A)} - y) = a(-y) - S_{\sigma(A)} + o(1)$, whence

$$\begin{aligned} g_A(x, y) - g_A^+(x) &= -w_A(x) - a(x - y) + E_x[a(S_{\sigma(A)} - y)] \\ &= a(-y) - a(x - y) - \frac{x}{\sigma^2} + o(1), \end{aligned} \quad (3.5)$$

where $o(1) \rightarrow 0$ as $y \rightarrow +\infty$ uniformly in $x \in \mathbb{Z}$. In Section 7 ((7.8)) we shall see that for $|x| \leq y$,

$$\left| a(-y) - a(x - y) - \frac{x}{\sigma^2} \right| \leq C \left(a(-|x|) - \frac{|x|}{\sigma^2} \right); \quad (3.6)$$

in particular the right side of (3.5) is $(1 + |x|) \times o(1)$ as $y \rightarrow \infty$ uniformly for $|x| \leq y$. This bound is presented here because of its bearing close relevance to (3.5), although we shall use it only to show Lemma 3.4 given at the end of this subsection and before proceeding to it we introduce some results that are used in the proof of (7.8) as well as Lemma 3.4.

Put

$$r_A^+ = 1 + \max A \quad \text{and} \quad r_A^- = -1 + \min A,$$

so that $A \subset [r_A^- + 1, r_A^+ - 1]$, and bring in the following subsets of \mathbb{Z} :

$$V^+ = \{x : \exists n \geq 1, p_A^n(x, r_A^+) > 0\}, \quad V^- = \{x \in \mathbb{Z} : \exists n \geq 1, p_A^n(x, r_A^-) > 0\};$$

$$\widehat{V}^+ = \{y : \exists n \geq 1, p_A^n(r_A^+, y) > 0\}, \quad \widehat{V}^- = \{y \in \mathbb{Z} : \exists n \geq 1, p_A^n(r_A^-, y) > 0\};$$

in other words, V^+ is the set of those points x from which the walk can enter into $[r_A^+, \infty)$ with a positive probability, and similarly for V_A^- and \widehat{V}_A^\pm .

Clearly $g_A^+(x) > 0$ for $x \geq r_A^+$. By (1.12)

$$g_A^+(x) = \sum_{z \in \mathbb{Z}} p_A^n(x, z) g_A^+(z) \quad \text{for all } x \in \mathbb{Z}, n \geq 1; \quad (3.7)$$

hence $g_A^+(x) > 0$ for $x \in V^+$, while if $x \notin V^+$, then $g_A(x, y) = 0$ for $y \geq r_A^+$, hence $g^+(x) = 0$; and similarly for g_A^- . Thus

$$V^\pm = \{x : g_A^\pm(x) > 0\} \quad \text{and} \quad \widehat{V}^\pm = \{y : g_{-A}^\mp(-y) > 0\}. \quad (3.8)$$

Write $H_B^x(z)$ for $P_x[S_{\sigma(B)} = z]$ for $B \subset \mathbb{Z}$. By the strong Markov property

$$g_A(x, y) = \sum_{z \notin A, z \leq r_A^+} H_{(-\infty, r_A^+]}^x(z) g_A(z, y), \quad x > r_A^+, y < r_A^-,$$

and, on letting $y \rightarrow -\infty$,

$$g_A^-(x) = \sum_{z \notin A, z \leq r_A^+} H_{(-\infty, r_A^+]}^x(z) g_A^-(z)$$

as in the same way for (2.15). Further let $x \rightarrow \infty$ to see that

$$C_A^+/\sigma^2 = \lim_{x \rightarrow \infty} g_A^-(x) = \sum_{z \notin A, z \leq r_A^+} H_{(-\infty, r_A^+]}^{+\infty}(z) g_A^-(z) (\leq \infty),$$

where the first equality is by definition and the second follows by (2.9) i.e., $H_{(-\infty, 0]}^x(z) \leq CH_{(-\infty, 0]}^\infty(z)$ when the last sum is finite and by Fatou's lemma when it is infinity. Taking the successive limits in the reverse order we can see that $\hat{g}_A^+(y) = g_{-A}^-(y)$ converges to the same sum as $y \rightarrow -\infty$ (the justification is slightly different but the result is obtained in another way in Lemma 7.1 and we do not give any detail of it). As a consequence we find

$$C_A^+ = C_{-A}^+ = \sigma^2 \sum_{z \notin A, z \leq r_A^+} H_{(-\infty, r_A^+]}^{+\infty}(z) g_A^-(z). \quad (3.9)$$

Plainly $H_{(-\infty, r_A^+]}^{+\infty}(r_A^+) > 0$ and these identities together show

Lemma 3.2. $C_A^+ = C_{-A}^+$ and the following three conditions are equivalent one another:

$$(a) \quad C_A^+ > 0; \quad (b) \quad g_A^-(r_A^+) > 0; \quad (c) \quad g_{-A}^-(r_A^-) > 0.$$

Lemma 3.3. *There exists a positive constant c such that*

$$g_A^+(x) g_{-A}^-(y) + g_A^-(x) g_{-A}^+(y) > c(1 + |x| + |y|) \quad (3.10)$$

for all x, y for which the left side is positive.

Proof. If both x and y remain in a finite set, the assertion is trivial. For reason of symmetry we suppose $x > r_A^+ \vee 1$. Further we may suppose $y < -r_A^-$, otherwise the left side of (3.10) being asymptotic to $4xy/\sigma^4$. If $g_A^-(r_A^+) = 0$, then by Lemma 3.2 $g_{-A}^-(r_A^-) = 0$, hence the left side of (3.10) vanishes whenever $x > r_A^+$ and $y < -r_A^-$. It remains to consider the case $g_A^-(r_A^+) > 0$. In this case by Lemma 3.2 $g_{-A}^-(y) \geq c_1$ ($y < -r_A^-$) for some c_1 so that $g_A^+(x) g_{-A}^-(y) \geq c_1' x$. Similarly $g_A^-(x) g_{-A}^+(y) \geq c_2' |y|$. \square

Lemma 3.4. *For each $M > 1$, uniformly for $-M < x \leq y$, as $y \rightarrow +\infty$*

$$g_A(x, y) = g_A^+(x) \{1 + o(1)\};$$

and for $x > 0$ and $y > \max A$,

$$g_A(-x, y) \leq g_A^+(-x) \{1 + o_y(1)\}, \quad (3.11)$$

where $o_y(1)$ is bounded and, as $y \rightarrow \infty$, tends to zero uniformly in x ; in (3.11) the equality holds if and only if $E[X^3; X > 0] < \infty$.

Proof. $g_A(x, y)$ is positive for all $y \geq r_A^+$ or zero for all $y \geq r_A^+$ according as $g_A^+(x) > 0$ or 0 . Since $g_A^+(x) \sim 2x/\sigma^2$ as $x \rightarrow \infty$, (3.5) and (3.6) show the first relation.

As for the second one we may suppose that $0 \in A$ and $g_A^+(r_A^-) > 0$, and let $\xi_0 = 0$ in (3.3). According to (the dual assertion of) Lemma 3.2, it follows from the latter condition that $g_A^+(-x)$ is bounded away from zero for $x \geq r_A^-$. Put $\hat{\lambda}(x) = a(-x) - x/\sigma^2$. Then (3.6) becomes

$$g_A(-x, y) - g_A^+(-x) = -\hat{\lambda}(x+y) + \hat{\lambda}(y) + o(1),$$

where $o(1) \rightarrow 0$ as $y \rightarrow \infty$ uniformly in $x \in \mathbb{Z}$, and since $g_A^+(-x) = \hat{\lambda}(x) + O(1)$, an application of (7.5) concludes the second relation. If $E[X^3; X > 0] < \infty$ then $\sigma^2 a(y) - y$ converges to a finite limit as $y \rightarrow \infty$, and by (3.5) it is easy to see that $g_A(-x, y) - g_A^+(-x)$ tends to zero as $x \wedge y \rightarrow \infty$ so that in (3.11) the inequality sign may be replaced by the equality sign, whereas if $E[X^3; X > 0] = \infty$, $g_A(-x, y)/g_A^+(-x)$ is made arbitrarily small by choosing an x large enough. The proof is complete. \square

3.2 Probabilities of escape from A and an overshoot estimate

What are advanced in this subsection are modifications of the corresponding results for the case $A = \{0\}$ that are given in [12, Section 2.3] and the arguments are parallel to those in it.

We have the identity

$$P_x[\sigma_{\{y\}} < \sigma_A] = \frac{g_A(x, y)}{g_A(y, y)}$$

whenever $y \neq x$. From (3.1) and (3.2) we deduce

$$g_A(y, y) = a(y) + a(-y) + O(1).$$

Hence, from Lemma 3.4 it follows that uniformly for $-M < x < y$,

$$P_x[\sigma_{\{y\}} < \sigma_A] = \frac{\sigma^2 g_A^+(x)}{2y} \{1 + o(1)\} \quad \text{as } y \rightarrow +\infty \quad (3.12)$$

and similarly for the case when $y \rightarrow -\infty$ and $y < x < M$ with $g_A^-(x)$ replacing $g_A^+(x)$ on the right side. In what follows the letter R will always denote a positive integer.

Proposition 2. *Uniformly for $x < R$,*

$$P_x[\sigma_{[R, \infty)} < \sigma_A] = P_x[\sigma_{\{R\}} < \sigma_A] \{1 + o(1)\} \quad \text{as } R \rightarrow +\infty.$$

And furthermore, for each $M > 1$, as $R \rightarrow \infty$, uniformly for $-M < x < R$,

$$P_x[\sigma_{[R, \infty)} < \sigma_A] = \frac{\sigma^2 g_A^+(x)}{2R} \{1 + o(1)\} \quad (3.13)$$

and

$$P_x[\sigma_{[R, \infty)} < \sigma_A] \leq \frac{\sigma^2 g_A^+(x)}{2R} \{1 + o(1)\} \quad \text{uniformly for } x < 0. \quad (3.14)$$

Proof. The difference $P_x[\sigma_{[R, \infty)} < \sigma_A] - P_x[\sigma_{\{R\}} < \sigma_A]$ is expressed as

$$\sum_{z > R} P_x[\sigma_{[R, \infty)} < \sigma_A, S_{\sigma([R, \infty))} = z] P_z[\sigma_{\{R\}} > \sigma_A]. \quad (3.15)$$

For simplicity suppose $A \subset (-\infty, 0]$. Then this sum is dominated by

$$P_x[\sigma_{[R,\infty)} < \sigma_A] \sup_{z > R} P_z[\sigma_{\{R\}} > \sigma_{(-\infty, 0]}],$$

of which the supremum tends to zero as $R \rightarrow \infty$ as is proved in [12, Lemma 2.3]. Thus we obtain the first relation of the proposition.

The second relation (3.13) follows immediately from the first and (3.12). Similarly (3.14) follows by using (3.11), the second half of Lemma 3.4 \square

Proposition 3. (*Overshoot estimate*) For each $M > 1$, uniformly for $-M \leq x < R$, as $R \rightarrow \infty$

$$\frac{1}{R} E_x[S_{\sigma([R,\infty))} - R | \sigma_{[R,\infty)} < \sigma_A] = o(1) \quad (3.16)$$

and for all $x \in \mathbb{Z}$ and $\xi_0 \in A$,

$$E_x[S_{\sigma([R,\infty))}; \sigma_{[R,\infty)} < \sigma_A] \leq \frac{1}{2}[\sigma^2 a(x - \xi_0) + x - \xi_0] + c_A \quad (3.17)$$

for some constant $c_A (\leq \frac{1}{2}\sigma^2 + \frac{1}{2} \sup_{\xi \in A}[\sigma^2 a(\xi - \xi_0) + \xi - \xi_0])$.

Proof. Suppose $0 \in A$ and $\xi_0 = 0$, which gives rise to no loss of generality since R does not appear in the right side of (3.17). (3.3) is then reduced to $g_A^+(y) = a^\dagger(y) + \sigma^{-2}y - E_y[a(S_{\sigma(A)}) + \sigma^{-2}S_{\sigma(A)}]$. That g_A^+ is non-negative and harmonic on $\mathbb{Z} \setminus A$ in the sense of the identity (1.12) implies that for all $x \in \mathbb{Z}$,

$$E_x[g_A^+(S_{\sigma([R,\infty))}); \sigma_{[R,\infty)} < \sigma_A] \leq g_A^+(x),$$

which shows (3.17), because $y \leq \frac{1}{2}(a(y) + \sigma^{-2}y) \leq \frac{1}{2}[\sigma^2 g_A^+(y) + \sup_{\xi \in A}(\sigma^2 a(\xi) + \xi)]$ (for all $y \in \mathbb{Z}$) and $\sigma^2 g_A^+(x) \leq \sigma^2[1 + a(x)] + x$. For R so large that $g_A^+(z)$ is monotone in $z \geq R$, it also follows that

$$\begin{aligned} 0 &\leq E_x[g_A^+(S_{\sigma([R,\infty))}) - g_A^+(R); \sigma_{[R,\infty)} < \sigma_A] \\ &\leq g_A^+(x) - g_A^+(R) P_x[\sigma_{[R,\infty)} < \sigma_A]. \end{aligned} \quad (3.18)$$

Owing to Proposition 2 the last member is dominated by

$$g_A^+(x) \left(1 - \frac{g_A^+(R)}{2R/\sigma^2}\right) + (1 \vee x) \times o(1) = (1 \vee x) \times o(1),$$

where $o(1) \rightarrow 0$ as $R \rightarrow \infty$ uniformly for $-M \leq x < R$. Dividing by $R P_x[\sigma_{[R,\infty)} < \sigma_A]$ and applying Proposition 2 again we find

$$\frac{1}{R} E_x[g_A^+(S_{\sigma([R,\infty))}) - g_A^+(R) | \sigma_{[R,\infty)} < \sigma_A] = o(1).$$

Finally substitution from $R = \frac{1}{2}\sigma^2 g_A^+(R)\{1 + o(1)\}$ ($R \rightarrow +\infty$) yields the formula (3.16). \square

By Markov's inequality we deduce from Propositions 2 and 3 the following

Corollary 3. Uniformly for $-M < x < R$ and for $R' > R$, as $R \rightarrow \infty$

$$\begin{aligned} P_x[S_{\sigma([R,\infty))} \geq R', \sigma_{[R,\infty)} < \sigma_A] &= \frac{R P_x[\sigma_{[R,\infty)} < \sigma_A]}{R' - R} \times o(1) \\ &= \frac{g_A^+(x)}{R' - R} \times o(1). \end{aligned} \quad (3.19)$$

Corollary 4. For any $\varepsilon > 0$, uniformly for $-M < x < R$, as $R \rightarrow \infty$

$$E_x[S_{\sigma([R,\infty))}; S_{\sigma([R,\infty))}] \geq (1 + \varepsilon)R, \sigma_{[R,\infty)} < \sigma_A] = g_A^+(x) \times o(1). \quad (3.20)$$

Proof. We have only to decompose $S_{\sigma([R,\infty))}$ into the sum of $S_{\sigma([R,\infty))} - R$ and R and to apply Propositions 3 and 2 for the first term and Corollary 3 for the second. \square

Proposition 4. For each $M > 1$, as $R \rightarrow \infty$

$$\begin{aligned} P_x[\tau_{U(R)} = \sigma_{[R,\infty)} < \sigma_A] &= P_x[\sigma_{[R,\infty)} < \sigma_A]\{1 + o(1)\} \\ &= \frac{\sigma^2 g_A^+(x)}{2R}\{1 + o(1)\}, \end{aligned}$$

where $o(1) \rightarrow 0$ uniformly for $-M \leq x \leq R$.

Proof. Given $\xi_0 \in A$, we put $f(z) = a(z - \xi_0) + \sigma^{-2}z$, so that in view of (3.3)

$$g_A^+(x) = \delta_{\xi_0, x} + f(x) - E_x[f(S_{\sigma_A})].$$

Suppose $x \neq \xi_0$, the case $x = \xi_0$ being readily reduced to this case (see (1.12)). In view of the optional sampling theorem $M_n = f(S_{n \wedge \sigma_A \wedge \tau_{U(R)}})$ is then a martingale under the law P_x . It is uniformly integrable and hence, according to the martingale convergence theorem,

$$f(x) = E_x[f(S_{\sigma_A \wedge \tau_{U(R)}})].$$

Breaking this expectation according as $\tau_{U(R)}$ is larger or smaller than σ_A we find

$$\begin{aligned} g_A^+(x) &= E_x[f(S_{\sigma_A \wedge \tau_{U(R)}})] - E_x[f(S_{\sigma_A})] \\ &= E_x[f(S_{\tau_{U(R)}}); \tau_{U(R)} < \sigma_A] - E_x[f(S_{\sigma_A}); \tau_{U(R)} < \sigma_A]. \end{aligned}$$

The last expectation uniformly converges to zero as $R \rightarrow \infty$. In view of Proposition 2 and the relation $f(R) = 2R/\sigma^2\{1 + o(1)\}$, for the proof of the proposition it therefore suffices to show

$$E_x[f(S_{\tau_{U(R)}}); \tau_{U(R)} < \sigma_A] - f(R)P_x[\sigma_{[R,\infty)} = \tau_{U(R)} < \sigma_A] = (1 \vee |x|) \times o(1) \quad (3.21)$$

as $R \rightarrow \infty$ uniformly for $-M < x < R$. By the dual relation of (3.17) we deduce

$$\begin{aligned} E_x[f(S_{\tau_{U(R)}}); \sigma_{(-\infty, -R]} = \tau_{U(R)} < \sigma_A] &\leq \left[\sup_{y \leq -R} \frac{f(y)}{-y} \right] E_x[|S_{\sigma_{(-\infty, -R]}}|; \sigma_{(-\infty, -R]} < \sigma_A] \\ &\leq [\sigma^2 a(-x) - x + 1] \times o(1) \end{aligned}$$

(uniformly for all x). On the other hand, owing to the overshoot estimate in (3.16)

$$E_x[f(S_{\tau_{U(R)}}) - f(R); \sigma_{[R,\infty)} = \tau_{U(R)} < \sigma_A] = (x \vee 1) \times o(1)$$

uniformly for $-M < x < R$. Adding these relations we obtain (3.21). \square

Corollary 5. For any $M > 1$, uniformly for $-M < x \leq R$ and $z \geq R$, as $R \rightarrow \infty$

$$\begin{aligned} P_x[S_{\sigma_{[R,\infty)}} = z | \tau_{U(R)} = \sigma_{[R,\infty)} < \sigma_A] \\ \leq P_x[S_{\sigma_{[R,\infty)}} = z | \sigma_{[R,\infty)} < \sigma_A]\{1 + o(1)\}. \end{aligned} \quad (3.22)$$

Proof. Denote the conditional probabilities on the left and on the right of (3.22) by $\mu(z)$ and $\tilde{\mu}(z)$, respectively. Then we see

$$\begin{aligned}\mu(z) &\leq \frac{P_x[\sigma_{[R,\infty)} < \sigma_A, S_{\sigma_{[R,\infty)}} = z]}{P_x[\tau_{U(R)} = \sigma_{[R,\infty)} < \sigma_A]} \\ &= \tilde{\mu}(z) \frac{P_x[\sigma_{[R,\infty)} < \sigma_A]}{P_x[\tau_{U(R)} = \sigma_{[R,\infty)} < \sigma_A]}.\end{aligned}$$

According to Proposition 4 this yields $\mu(z) \leq \tilde{\mu}(z)\{1 + o(1)\}$, as desired. \square

The next result concerns the probability of the walk escaping from A , which, though not applied in this paper, we record here.

Corollary 6. *Uniformly for $|x| < R$, as $R \rightarrow \infty$*

$$P_x[\tau_{U(R)} < \sigma_A] = \frac{\sigma^2 u_A(x)}{R} \{1 + o(1)\}.$$

Proof. The inclusion-exclusion formula derives the assertion of the proposition from Proposition 4 if $P_x[\sigma_{(-\infty, -R]} \vee \sigma_{[R, \infty)} < \sigma_{\{0\}}] = o(x/R)$ uniformly for $0 < |x| < R$, which is readily verified (cf. [12, Proposition 2.4]). \square

4 Results for a single point set and a half line

Put

$$\mathbf{g}_t(u) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-u^2/2\sigma^2 t} \quad (u \in \mathbb{R}, t > 0). \quad (4.1)$$

We shall apply the following version of local limit theorem (cf. e.g., [10], [17]): as $n \rightarrow \infty$, uniformly for $x \in \mathbb{Z}$ with $p^n(x) > 0$,

$$p^n(x) = \nu \mathbf{g}_n(x) + o\left(\frac{\sqrt{n}}{n \vee x^2}\right), \quad (4.2)$$

where ν designates the (temporal) period of the walk: $\nu = \text{g.c.d. of } \{n : p^n(0) > 0\}$.

4.1 Results for the case $A = \{0\}$

Here we state two results from [12] and [11] for the case $A = \{0\}$ that are used later.

Theorem A. *Given a constant $M > 1$, the following asymptotic estimates of $p_{\{0\}}^n(x, y)$ as $n \rightarrow \infty$, stated in three cases of constraints on x and y , hold true uniformly for x and y subject to the respective constraints.*

(i) *Under $|x| \vee |y| < M\sqrt{n}$ and $|x| \wedge |y| = o(\sqrt{n})$,*

$$p_{\{0\}}^n(x, y) = \frac{\sigma^4 a^\dagger(x) a(-y) + xy}{\sigma^2 n} p^n(y - x) + o\left(\frac{(|x| \vee 1)|y|}{n^{3/2}}\right). \quad (4.3)$$

(ii) Under $M^{-1}\sqrt{n} < |x|, |y| < M\sqrt{n}$ (both $|x|$ and $|y|$ are between the two extremes),

$$p_{\{0\}}^n(x, y) = \nu \left[\mathbf{g}_n(y - x) - \mathbf{g}_n(y + x) \right] + o\left(\frac{1}{\sqrt{n}}\right) \quad \text{if } xy > 0 \text{ and } p^n(y - x) > 0, \quad (4.4)$$

$$p_{\{0\}}^n(x, y) = o\left(\frac{1}{\sqrt{n}}\right) \quad \text{if } xy < 0. \quad (4.5)$$

(iii) Let $1 \leq |x| \wedge |y| < \sqrt{n} < |x| \vee |y|$. Then, if $E|X|^{2+\delta} < \infty$ for some $\delta \geq 0$,

$$p_{\{0\}}^n(x, y) = O\left(\frac{|x| \wedge |y|}{|x| \vee |y|} \mathbf{g}_{4n}(|x| \vee |y|)\right) + \frac{|x| \wedge |y|}{(|x| \vee |y|)^{2+\delta}} \times o(1). \quad (4.6)$$

Theorem B. Uniformly in x , as $n \rightarrow \infty$

$$P_x[\sigma_{\{0\}} = n] = \frac{\sigma^2 a^\dagger(x)}{n} p^n(-x) + \frac{|x| \vee 1}{n^{3/2} \vee |x|^3} \times o(1).$$

Theorem A is Theorem 1.1 of [12]. Theorem B is an immediate consequence of Corollary 1.1 of [11] and the local limit theorem (4.2), if $\nu = 1$, i.e., if the walk S is aperiodic. For $\nu > 1$, apply this result to the walk $\tilde{S}_n = \nu^{-1} S_{\nu n}$, whose increment has variance σ^2/ν .

From Theorem A it follows that as $|x| \wedge |y| \rightarrow \infty$

$$\begin{cases} \text{(a)} & p_{\{0\}}^n(x, y) \asymp |xy|n^{-3/2} & (xy > 0, |x| \vee |y| = O(\sqrt{n})), \\ \text{(b)} & p_{\{0\}}^n(x, y) = o(|xy|n^{-3/2}) & (xy < 0). \end{cases} \quad (4.7)$$

Indeed, (a) is obvious, whereas for any M , (b) follows for $|x| \vee |y| < M\sqrt{n}$ by (i) on one hand and $p_{\{0\}}^n(x, y) < C|xy|n^{-3/2}M^{-2}$ for $|x| \vee |y| \geq M\sqrt{n}$ by (ii) on the other hand.

Proof of Proposition 1. The bound of the proposition follows from the local limit theorem if $|x| \wedge |y| \geq \sqrt{n}$, from (4.6) if $|x| \wedge |y| < \sqrt{n} \leq |x| \vee |y|$, and from (4.3) and (4.7) if $|x| \vee |y| < \sqrt{n}$. \square

4.2 Space-time distribution of entrance into $(-\infty, 0]$

Here we consider the walk killed when it enters $(-\infty, 0]$. The results given in this subsection will be used only in Section 6 where the case $xy < 0$ is intensively studied.

The following result is obtained as a special case of [4, Proposition 11] that concerns asymptotically stable random walks (cf. also [12, Theorem 1.3]).

Theorem C. For each $M > 1$, uniformly for $0 < x, y \leq M\sqrt{n}$, as $xy/n \rightarrow 0$

$$p_{(-\infty, 0]}^n(x, y) = \frac{2f^+(x)f^-(y)}{\sigma^2 n} p^n(y - x) \{1 + o(1)\}.$$

In the regime $M^{-1}\sqrt{n} < x, y < M\sqrt{n}$ excluded from this theorem $p_{(-\infty, 0]}^n(x, y)$ has the same asymptotic form as $p_{\{0\}}^n(x)$ which is given in Theorem A (ii), namely

$$p_{(-\infty, 0]}^n(x, y) = \nu \left[\mathbf{g}_n(y - x) - \mathbf{g}_n(y + x) \right] + o\left(\frac{1}{\sqrt{n}}\right) \quad (4.8)$$

provided $p^n(y - x) > 0$ (see Remark 4 after Lemma 5.1 in the next section).

From Theorem C we derive an asymptotic form of the space-time distribution of the first entrance into $(-\infty, 0]$, which we denote by $h_x(n, y)$: for $y \leq 0$

$$h_x(n, y) = P_x[S_{\sigma_{(-\infty, 0]}} = y, \sigma_{(-\infty, 0]} = n].$$

We suppose a as well as $H_{(-\infty, 0]}^\infty$ to be extended to continuous variables by linear interpolation for notational convenience.

Lemma 4.1. *Define $\alpha_n(x, y)$ for $y \leq 0 < x$ via the equation*

$$h_x(n, y) = \frac{\nu f^+(x) \mathbf{g}_n(x)}{n} \left[H_{(-\infty, 0]}^{+\infty}(y) + \alpha_n(x, y) \right] \quad (4.9)$$

if $p^n(y - x) > 0$, and put $\alpha_n(x, y) = 0$ if $p^n(y - x) = 0$. Then for each $\varepsilon > 0$ and $M \geq 1$, $\alpha^n(x, y)$ can be decomposed as

$$\alpha_n(x, y) = \beta_{n, \varepsilon}(x, y) + H_{(-\infty, 0]}^{+\infty}(y) \times o_{\varepsilon, M}(1),$$

where $o_{\varepsilon, M}(1)$ is bounded and, as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ in this order, approaches zero uniformly for $0 < x < M\sqrt{n}$ and $y < 0$, and the function of $y \leq 0$ defined by

$$\beta_{n, \varepsilon}(y) := \sup_{n/2 \leq k \leq n} \sup_{0 < x < M\sqrt{n}} |\beta_{k, \varepsilon}(x, y)|,$$

satisfies

$$\beta_{n, \varepsilon}(y) \leq c_M H_{(-\infty, 0]}^{+\infty}(y), \quad \lim_{n \rightarrow \infty} \sum_{y=-\infty}^0 \beta_{n, \varepsilon}(y) = 0 \quad (4.10)$$

and

$$\sum_{z=-\infty}^0 \beta_{n, \varepsilon}(z) (|z| \vee |y|) \leq C|y| \frac{\sigma^2 a(\varepsilon\sqrt{n}) - \varepsilon\sqrt{n}}{\varepsilon\sqrt{n}}. \quad (4.11)$$

Proof. Suppose the walk is aperiodic for simplicity. We have the representation

$$h_x(n, y) = \sum_{w=1}^{\infty} p_{(-\infty, 0]}^{n-1}(x, w) p(y - w). \quad (4.12)$$

In view of Theorem C and the local limit theorem (4.2), for each $\varepsilon > 0$, the sum on the right over $w < 4\varepsilon\sqrt{n}$ may be replaced by

$$\frac{f^+(x) \mathbf{g}_n(x)}{n} \cdot \frac{2}{\sigma^2} \sum_{1 \leq w \leq 4\varepsilon\sqrt{n}} f^-(w) p(y - w) (1 + o_{\varepsilon, M}(1)).$$

By definition $\alpha_n(x, y)$ is then expressed as

$$\alpha_n(x, y) = -\frac{2}{\sigma^2} \sum_{1 \leq w \leq 4\varepsilon\sqrt{n}} f^-(w) p(y - w) \times o_{\varepsilon, M}(1) + \beta_{n, \varepsilon}(x, y), \quad (4.13)$$

where $\beta_{n,\varepsilon}(x, y)$ is defined to be

$$-\frac{2}{\sigma^2} \sum_{w>4\varepsilon\sqrt{n}} f^-(w)p(y-w) + \left(\frac{f^+(x)\mathbf{g}_n(x)}{n} \right)^{-1} \sum_{w>4\varepsilon\sqrt{n}} p_{(-\infty,0]}^{n-1}(x, w)p(y-w).$$

By Proposition 1 $p_{(-\infty,0]}^{n-1}(x, w) \leq xwn^{-3/2}$ and we infer that

$$|\beta_{n,\varepsilon}(x, y)| \leq C \sum_{w>4\varepsilon\sqrt{n}} wp(y-w),$$

where C may depend on M . The first term on the right side of (4.13) may be replaced by $H_{(-\infty,0]}^{+\infty}(y) \times o_{\varepsilon,M}(1)$ and the verification of (4.10) is now immediate.

For the proof of (4.11) observe first that by the change of variable $w = 4\varepsilon\sqrt{n} + z$

$$|\beta_{n,\varepsilon}(x, y)| \leq C_1 H_{(-\infty,0]}^{+\infty}(y - 4\varepsilon\sqrt{n}) + C_2 \varepsilon \sqrt{n} F(y - 4\varepsilon\sqrt{n}),$$

and then that on summing by parts and abbreviating $H_{(-\infty,0]}^{+\infty}$ to H ,

$$\begin{aligned} \sum_{z=-\infty}^0 (|z| \wedge |y|) |\beta_{n,\varepsilon}(x, z)| &= \sum_{z=y}^{-1} \sum_{w=-\infty}^z |\beta_{n,\varepsilon}(x, w)| \\ &\leq C_3 \sum_{z=y}^{-1} \left[\sum_{w=-\infty}^{z-4\varepsilon\sqrt{n}} H(w) + 4\varepsilon\sqrt{n} H(z - 4\varepsilon\sqrt{n}) \right] \\ &\leq C_3 |y| \left[\sum_{w=-\infty}^{y-4\varepsilon\sqrt{n}} H(w) + 4\varepsilon\sqrt{n} H(-4\varepsilon\sqrt{n}) \right] \end{aligned}$$

where (2.7) is applied for the first inequality and the monotonicity of $H(w)$ for the second. Since $4\varepsilon\sqrt{n} H(-4\varepsilon\sqrt{n}) \leq 8 \sum_{w \leq -2\varepsilon\sqrt{n}} H(w)$ and since

$$\sum_{w \leq t} H(w) \leq C'' \sum_{w \leq t} H(w) g(w, t) / |t| \leq C'' [a(-t) + t/\sigma^2] / |t| \quad (t < 0),$$

we conclude

$$\sum_{z=-\infty}^0 (|z| \wedge |y|) |\beta_{n,\varepsilon}(x, z)| \leq C |y| \frac{a(-2\varepsilon\sqrt{n}) - 2\varepsilon\sqrt{n}}{2\varepsilon\sqrt{n}}.$$

This bound is valid with $\beta_{\varepsilon,n}(z)$ in place of $|\beta_{n,\varepsilon}(x, z)|$ as easily checked by following the above derivation with k instead of n . Thus (4.11) is proved. \square

From the cheaper estimate (4.10) we have the asymptotic form of the distribution of hitting time of $(-\infty, 0]$. We record it here for later citations: as $n \rightarrow \infty$

$$\frac{1}{\nu} \sum_{j=0}^{\nu-1} P_x[\sigma_{(-\infty,0]} = n + j] \sim \frac{f^+(x)\mathbf{g}_n(x)}{n} \quad (4.14)$$

uniformly for $0 < x < M\sqrt{n}$ (cf. [12, Corollary 1.1] for an estimate for $x > \sqrt{n}$).

5 Proof of Theorem 1

We break Theorem 1 into three assertions by dividing the range of variables into three regimes according as $|x| \wedge |y| \rightarrow \infty$, $|x| \vee |y| \rightarrow \infty$ or $|x| \vee |y| = O(1)$, of which the first case is dealt with by Lemma 5.1, the second by Lemma 5.2 and the third in the ‘*Proof of Theorem 1*’ given at the end of this section.

Lemma 5.1. *As $x \wedge y \wedge n \rightarrow \infty$ under the constraint $x \vee y < M\sqrt{n}$,*

$$p_{\{0\}}^n(x, y) - p_A^n(x, y) = p_{\{0\}}^n(x, y) \times O\left(\frac{1}{x \wedge y}\right) \quad (5.1)$$

and $p_A^n(-x, y) = o(xyn^{-3/2})$.

Proof. The second relation is (4.7 b). For the proof of (5.1) suppose $0 \in A$ for simplicity—otherwise replace $\{0\}$ by $\{c\}$ with $c \in A$ —so that

$$p_{\{0\}}^n(x, y) - p_A^n(x, y) = \sum_{k=0}^n \sum_{\xi \in A \setminus \{0\}} P_x[\sigma_A = k, S_k = \xi] p_{\{0\}}^{n-k}(\xi, y). \quad (5.2)$$

We split the outer sum at $k = \lfloor n/2 \rfloor$, and denote the double sum restricted to $k < n/2$ by $I_{[0, n/2]}$ and the other part of the sum by $I_{[n/2, n]}$. According to Theorem A (i)

$$I_{[0, n/2]} \leq \sharp A \sup_{k < n/2} \sup_{\xi \in A} p_{\{0\}}^{n-k}(\xi, y) \leq C \frac{y}{n} p^n(y),$$

where $\sharp A$ denotes the cardinality of A . In a similar way, applying Theorem B to find that

$$I_{[n/2, n]} \leq \frac{Cx}{n^{3/2}} \sup_{\xi} \sum_{n/2 \leq k \leq n} p_{\{0\}}^{n-k}(\xi, y) \leq \frac{C'x}{n^{3/2}}.$$

where it is used for the second inequality that $\sum_{j=1}^{\infty} p_{\{0\}}^j(\xi, y) = g_{\{0\}}(\xi, y) \leq C''$ ($\xi \leq \max A$, $y > 0$). Adding these two bounds concludes the proof of Lemma 5.1. \square

REMARK 4. With minor modification the above proof also shows

$$p_{\{0\}}^n(x, y) - p_{(-\infty, 0]}^n(x, y) = p_{(-\infty, 0]}^n(x, y) \times [O(1/y) + o(\lambda(x)/x)]. \quad (5.3)$$

Indeed the evaluation of $I_{[n/2, n]}$ made in it applies to the case $A = (-\infty, 0]$ if we use (4.14) instead of Theorem B, while for $I_{[0, n/2]}$ we apply (4.7 b), (2.15) and Theorem C in turn to obtain

$$I_{[0, n/2]} = \sum_{\xi \leq 0} H_{(-\infty, 0]}^x(\xi) \times o\left(\frac{|\xi|y}{n^{3/2}}\right) = o\left(\frac{\lambda(x)y}{n^{3/2}}\right) = p_{(-\infty, 0]}^n(x, y) \times o\left(\frac{\lambda(x)}{x}\right).$$

Although (5.3) is not used in this paper (apart from (4.8)), it indicates the possibility for another approach based on the results for the walks killed on half lines.

In view of Theorem A and the local limit theorem (4.2) it follows from Lemma 5.1 that as $x \wedge y \wedge n \rightarrow \infty$ under the constraint $x \vee y < M\sqrt{n}$,

$$p_A^n(x, y) = \nu[\mathbf{g}_n(y - x) - \mathbf{g}_n(x + y)](1 + o(1)), \quad (5.4)$$

provided $p^n(y - x) > 0$. This proves the second half of Theorem 1.

Lemma 5.2. For any $M \geq 1$, uniformly for $-M < x < \sqrt{n}/\lg n$, as $y \rightarrow \infty$ under $y < M\sqrt{n}$,

$$p_A^n(x, y) = \frac{g_A^+(x)y}{n} p^n(y-x) \{1 + o(1)\};$$

and uniformly for $-M < y < \sqrt{n}/\lg n$, as $x \rightarrow +\infty$ under $x < M\sqrt{n}$,

$$p_A^n(x, y) = \frac{xg_{-A}^+(-y)}{n} p^n(y-x) \{1 + o(1)\}.$$

Proof. We have only to prove the first relation, the second one being the dual of it. We may suppose $g_A^+(x) > 0$, otherwise $p_A^n(x, y) = 0$ for all large y . Put $R = \lfloor \sqrt{n}/\lg n \rfloor$ and $N = mR^2 \lfloor \lg n \rfloor$ with m determined shortly and decompose

$$\begin{aligned} p_A^n(x, y) &= \sum_{k=1}^{N-1} \sum_{|z| \geq R} P_x[\tau_{U(R)} = k < \sigma_A, S_k = z] p_A^{n-k}(z, y) \\ &\quad + \varepsilon(x, y; R). \end{aligned} \quad (5.5)$$

Here

$$\varepsilon(x, y; R) = \sum_z P_x[\tau_{U(R) \setminus A} \geq N, S_N = z] p_A^{n-N}(z, y) \quad (5.6)$$

and $\lfloor t \rfloor$ denotes the largest integer that does not exceed t . Using the central limit theorem we deduce $\sup_{x \in U(R)} P_x[\tau_{U(R)} > R^2] < c$ for all sufficiently large R with a universal constant $c < 1$, and hence that

$$\varepsilon(x, y; R) \leq C e^{-\lambda N/R^2} / \sqrt{n}$$

with $\lambda = -\lg c$. Now take $m = 2/\lambda$ so that

$$\varepsilon(x, y; R) = O(n^{-2}); \quad (5.7)$$

hence $\varepsilon(x, y; R)$ is negligible. (Note that $p_A^n(x, y) = 0$ if $g_A^+(x) = 0$.)

As for the double sum in (5.5) we first notice that the contribution from the half line $z \leq -R$ is negligible in comparison with that from $z \geq R$ owing to (3.14) and the second relation of Lemma 5.1. It therefore remains to evaluate

$$W := \sum_{k=1}^{N-1} \sum_{z \geq R} P_x[\tau_{U(R)} = k < \sigma_A, S_k = z] p_A^{n-k}(z, y) \quad (5.8)$$

so as to verify

$$W = \frac{g_A^+(x)y}{n} p^n(y-x) \{1 + o(1)\}. \quad (5.9)$$

From (5.1) and Theorem A (i) and (ii) of Section 4.1 it follows that if $k < N$,

$$p_A^{n-k}(z, y) \begin{cases} = \frac{2zy}{\sigma^2 n} p^n(y-z) \{1 + o(1)\} & \text{for } R \leq z < 2R, \\ \leq zy/n^{3/2} & \text{for } 2R \leq z < \sqrt{n}. \end{cases} \quad (5.10)$$

By the same reasoning as for (5.7) it also follows that

$$P_x[\sigma_{[R, \infty)} = \tau_{U(R)} < \sigma_A] = \sum_{k=1}^{N-1} \sum_{z \geq R} P_x[\tau_{U(R)} = k < \sigma_A, S_k = z] - O(n^{-2}).$$

From these observations we infer that

$$\begin{aligned}
& \left| W - P_x[\sigma_{[R,\infty)} = \tau_{U(R)} < \sigma_A] \frac{2Ry}{\sigma^2 n} p^n(y-x) \right| \\
& \leq \sum_{R \leq z < 2R} P_x[\tau_{U(R)} < \sigma_A, S_{\tau_{U(R)}} = z] \left| \frac{2zy}{\sigma^2 n} p^n(y-z)\{1+o(1)\} - \frac{2Ry}{\sigma^2 n} p^n(y-x) \right| \\
& \quad + \frac{y}{n^{3/2}} E_x[S_{\tau_{U(R)}}; \tau_{U(R)} < \sigma_A, S_{\tau_{U(R)}} \geq 2R] \\
& \quad + \frac{CRy}{n^{3/2}} P_x[\tau_{U(R)} < \sigma_A, S_{\tau_{U(R)}} \geq 2R] \\
& \quad + Cn^{-2}.
\end{aligned} \tag{5.11}$$

On the right side we may replace $\tau_{U(R)}$ by $\sigma_{[R,\infty)}$ for obvious reason. In view of the local limit theorem, $p^n(y-z) = p^n(y-x)\{1+o(1)\}$ uniformly for $|z| < 2R$, hence

$$|zp^n(y-z) - Rp^n(y-x)| \leq zp^n(y-x) \times o(1) + |z-R|/\sqrt{n},$$

provided $p^n(y-x)p^n(y-z) > 0$, and we infer that the first term on the right side of (5.11) is at most a constant multiple of

$$\sum_{R \leq z < 2R} P_x[\sigma_{[R,\infty)} < \sigma_A, S_k = z] \left[\frac{2zy}{\sigma^2 n} p^n(y-x) \times o(1) + \frac{2(z-R)y}{\sigma^2 n^{3/2}} \right],$$

which, on applying Propositions 3 and 2 in turn, is

$$P_x[\sigma_{[R,\infty)} < \sigma_A] \frac{Ry}{n^{3/2}} \times o(1) = \frac{yg_A^+(x)}{n^{3/2}} \times o(1).$$

By Corollary 4 (in Section 3.2) the second term on the right side of (5.11) is at most

$$\frac{yg_A^+(x)}{n^{3/2}} \times o(1).$$

Similarly by Corollary 3 the third term admits this same bound. Thus we see the difference on the left side of (5.11) is negligible. On the other hand by Proposition 4

$$P_z[\sigma_{[R,\infty)} = \tau_{U(R)} < \sigma_A] \frac{2Ry}{\sigma^2 n} p^n(y-x) = \frac{g_A^+(x)y}{n} p^n(y-x)\{1+o(1)\}.$$

Consequently we obtain (5.9) as required. \square

Proof of Theorem 1. Owing to Lemmas 5.1 and 5.2 we may and do suppose $|x| \vee |y| \leq M$ for a constant M (cf. Remark 1 (b)). We make the same argument (with the same R) as in the preceding proof except that in it y is supposed to tend to $+\infty$ and we have neglected the contribution of the sum over $z \leq -R$ to the double sum in (5.5) but here y remains bounded and the contributions both from $z \geq R$ and from $z \leq -R$ become relevant. We apply Lemma 5.2 to see that if $R \leq |z| < \sqrt{n}$, then in the double sum in (5.5),

$$p_A^{n-k}(z, y) = p_{-A}^{n-k}(-y, -z) = \begin{cases} \frac{g_{-A}^-(-y)z}{n} p^n(y-z)\{1+o(1)\} & \text{for } z \geq R, \\ \frac{g_{-A}^+(-y)(-z)}{n} p^n(y-z)\{1+o(1)\} & \text{for } z \leq -R; \end{cases}$$

also by Theorem A (iii), if $|z| \geq \sqrt{n}$, then $p_A^{n-k}(z, y) \leq C|y|/z^2$. The evaluation of the term $\varepsilon(n, x, y)$ given in (5.6) is valid and making estimation of the overshoots as in the last several lines of the preceding proof we deduce

$$\begin{aligned} p_A^n(x, y) &= P_x[\tau_{U(R)} = \sigma_{[R, \infty)} < \sigma_A] \frac{g_{-A}^-(-y)R}{n} p^n(y-x) \{1 + o(1)\} \\ &\quad + P_x[\tau_{U(R)} = \sigma_{(-\infty, -R]} < \sigma_A] \frac{g_{-A}^+(-y)R}{n} p^n(y-x) \{1 + o(1)\} \\ &\quad + O(n^{-2}). \end{aligned}$$

Substitution from the formula of Proposition 4 and its dual therefore concludes the required relation of Theorem 1. \square

REMARK 5. Corollary 1 is verified virtually in the proof above. For in it we may replace $p_A^{n-k}(z, y)$ by $\widehat{p}_A^{n-k}(z, y)$ with the difference when $y \in A$ in which case the latter represents the hitting distribution of A in space and time, i.e., $\widehat{p}_A^{n-k}(z, y) = P_z[\sigma_A = n, S_n = y]$. A direct derivation may also be made by substituting the expression of $p_A^{n-1}(x, y)$ given in Theorem 1 into the identity

$$P_x[\sigma_A = n, S_n = \xi] = \sum_{y \notin A} p_A^{n-1}(x, y) p(\xi - y),$$

(the last leaving decomposition) and then using the dual relations of (1.12) that read

$$\sum_{y \notin A} p(\xi - y) g_{-A}^\pm(-y) = g_{-A}^\pm(-\xi), \quad \xi \in A.$$

6 Refinements in case $xy < 0$

In the case $xy < 0$ the range of validity of formula (1.4) is restricted to $|x| \wedge |y| < M$ in Theorem 1. In this section we remove this restriction under additional conditions on p .

We call

$$D_A^+ = \lim_{x \rightarrow \infty} [\sigma^2 a(x) - x - \sigma^2 g_A^-(x)].$$

By (3.4) the limit exists and for all $\xi_0 \in A$,

$$D_A^+ = \sum_{\xi \in A} H_A^{+\infty}(\xi) [\sigma^2 a(\xi - \xi_0) - (\xi - \xi_0)]. \quad (6.1)$$

D_A^+ is positive unless either A consists of a single point or the walk S is left-continuous when what we discuss below becomes trivial.

Proposition 5. *For any $M > 1$ and $c \in \mathbb{Z}$, as $x \wedge (-y) \wedge n \rightarrow \infty$ under the condition $x \vee (-y) < M\sqrt{n}$*

$$p_{\{c\}}^n(x, y) - p_A^n(x, y) = \left(D_A^+ + o(1)\right) \frac{x + |y|}{\sigma^2 n} p^n(y-x); \quad (6.2)$$

for $c \in A$, in other words, the probability that the path of a pinned walk of length n joining x and y passes through A but avoids the point c is asymptotically equivalent to $D_A^+(x + |y|)/\sigma^2 n$.

Proof. Throughout the proof the variables x, y and n are assumed subject to the restriction $x \vee |y| < M\sqrt{n}$ and $y < 0 < x$.

We can suppose that $c \in A$. Indeed, if (6.2) is valid under this condition, taking any $b \notin A$ and applying (6.2) with $\{c, b\}$ in place of A , we have two equalities and the subtraction of them yields

$$p_{\{c\}}^n(x, y) - p_{\{b\}}^n(x, y) = (x + |y|) \times o(n^{-3/2}),$$

which, combined with the expression for $p_{\{c\}}^n - p_A^n$, shows (6.2) with b in place of c .

Now suppose $c \in A$. Then

$$p_{\{c\}}^n(x, y) - p_A^n(x, y) = \sum_{k=1}^n \sum_{\xi \in A \setminus \{c\}} P_x[\sigma_A = k, S_k = \xi] p_{\{c\}}^{n-k}(\xi, y). \quad (6.3)$$

Let ε be a positive number (small enough that $\varepsilon M^2 < 1/2$). For $\xi \in A$, by Corollary 1 and (1.14) uniformly for $k > \varepsilon x^2$, as $x \rightarrow \infty$

$$P_x[\sigma_A = k, S_k = \xi] = \frac{\sigma^2 a(x) + x}{2k} p^k(\xi - x) H_A^{+\infty}(\xi) \{1 + o(1)\}$$

whereas by Theorem 1 (or Theorem A) as $y \rightarrow -\infty$, uniformly for $0 \leq k \leq n - \varepsilon y^2$,

$$p_{\{c\}}^{n-k}(\xi, y) = \frac{g_{\{c\}}^-(\xi)}{n - k} |y| p^{n-k}(y - \xi) \{1 + o(1)\} \quad \text{for } \xi \in A.$$

Denote the outer sum in (6.3) restricted on an interval $a < k \leq b$ by $I_{(a,b]}$. We then obtain

$$\begin{aligned} & I_{(\varepsilon x^2, n - \varepsilon y^2]} \\ &= \sum_{\xi \in A \setminus \{c\}} H_A^{+\infty}(\xi) g_{\{c\}}^-(\xi) \sum_{\varepsilon x^2 < k \leq n - \varepsilon y^2} \frac{x p^k(\xi - x)}{k} \cdot \frac{|y| p^{n-k}(y - \xi)}{n - k} \{1 + o(1)\}. \end{aligned}$$

Suppose $p^n(y - x) > 0$. Then recalling that ν denotes the period of the walk, we apply the local limit theorem (4.2) to rewrite the inner sum as

$$\nu \int_{\varepsilon x^2}^{n - \varepsilon y^2} \frac{x \mathbf{g}_s(x)}{s} \cdot \frac{|y| \mathbf{g}_{n-s}(y)}{n - s} ds \{1 + o(1)\}.$$

Since $x \mathbf{g}_s(x)/s$ is the passage time density of Brownian motion, the integral above, if the range of integration is extended to the interval $(0, n)$, becomes

$$\frac{x + |y|}{n} \mathbf{g}_n(x + |y|) = \frac{x + |y|}{\nu n} p^n(y - x) \{1 + o(1)\}.$$

Plainly

$$\int_0^{\varepsilon x^2} x \mathbf{g}_s(x) s^{-1} ds = \int_0^\varepsilon \mathbf{g}_u(1) u^{-1} du = o(\varepsilon)$$

as $\varepsilon \downarrow 0$ and similarly $\int_{n - \varepsilon y^2}^n |y| \mathbf{g}_{n-s}(y) (t - s)^{-1} ds = o(\varepsilon)$, so that for ε small enough,

$$\left(\int_0^{\varepsilon x^2} + \int_{n - \varepsilon y^2}^n \right) \frac{x \mathbf{g}_s(x)}{s} \cdot \frac{|y| \mathbf{g}_{n-s}(y)}{n - s} ds \leq \frac{\varepsilon}{n} [|y| \mathbf{g}_n(y) + x \mathbf{g}_n(x)].$$

Since $g_{\{c\}}^-(\xi) = a(\xi - c) - (\xi - c)/\sigma^2$ for $\xi \neq c$, we also have

$$\sum_{\xi \in A \setminus \{c\}} H_A^{+\infty}(\xi) g_{\{c\}}^-(\xi) = D_A^+/\sigma^2.$$

We then put these together to obtain

$$I_{(\varepsilon x^2, n - \varepsilon y^2]} = D_A^+ \frac{x + |y|}{\sigma^2 n} p^n(y - x) \left[1 + o(1) + O(\varepsilon) \right], \quad (6.4)$$

where $o(1) \rightarrow 0$ as $x \wedge (-y) \wedge n \rightarrow \infty$ for each $\varepsilon > 0$ and $O(\varepsilon)$ is uniform in x, y and n .

As for the remaining parts of the double sum in (6.3) we observe

$$\begin{aligned} I_{(0, \varepsilon x^2]} &\leq CP_x[\sigma_A \leq \varepsilon x^2] p_{\{c\}}^n(\xi, y) \\ &\leq C' P_0[\max_{k \leq \varepsilon x^2} (-S_k) \geq x] \frac{|y|}{n^{3/2}} \leq C'' \frac{\varepsilon |y|}{n^{3/2}}, \end{aligned} \quad (6.5)$$

where Kolmogorov's inequality is used for the last inequality. From (i) and (iii) of Theorem A we derive

$$\begin{aligned} p_{\{0\}}^j(\xi, y) &\leq \frac{Cy}{j^{3/2}} e^{-y^2/2\sigma^2 j} && \text{for } j > y^2, \\ &\leq \frac{C}{y\sqrt{j}} e^{-y^2/8\sigma^2 j} + \frac{1}{y^2} \times o(1) && \text{for } 1 \leq j \leq y^2, \end{aligned}$$

where $o(1)$ is bounded and tends to zero as $j \rightarrow \infty$ uniformly in y , and on using these bounds easy computations yield

$$I_{(n - \varepsilon y^2, n]} \leq C \left[\sum_{1 \leq j \leq \varepsilon y^2} \frac{1}{|y|\sqrt{j}} e^{-y^2/8\sigma^2 j} \right] \frac{x}{n^{3/2}} < C' \frac{\varepsilon x}{n^{3/2}}.$$

This together with (6.5) and (6.4) completes the proof. \square

Because of the identity $C_A^+ = C^+ - D_A^+$ Theorem 2 follows immediately from Proposition 5 together with the result for the case $A = \{0\}$ that is established in [12, Theorem 1.4]. (For verification of the latter one can readily adapt the proof of Proposition 6 given shortly.)

Theorem 2 entails that (1.4), the formula of Theorem 1, is true uniformly under $|x| \vee |y| = O(\sqrt{n})$ and $|x| \wedge |y| = o(\sqrt{n})$, if $E[|X|^3; X < 0] < \infty$. The next result asserts that the same is true under a much weaker condition on p . We state the condition by means of a as follows: for $x, y \geq 1$,

$$\frac{(\sigma^2 a(x) - x)/x}{(\sigma^2 a(y) - y)/y} \rightarrow 0 \quad \text{as} \quad \frac{y}{x} \rightarrow 0. \quad (6.6)$$

Theorem 3. *Suppose (6.6) to hold true. Then, for any $M > 1$, uniformly for*

$$-M\sqrt{n} \leq y < 0 < x \leq M\sqrt{n}, \quad (6.7)$$

as $n \rightarrow \infty$ and $(x \wedge |y|)/\sqrt{n} \rightarrow 0$ formula (1.4) of Theorem 1 holds true; in particular if $E[|X|^3; X < 0] = \infty$ (so that $\sigma^2 a(x) - x \rightarrow \infty$ as $x \rightarrow \infty$), as $x \wedge |y| \wedge n \rightarrow \infty$ under (6.7) and $x \wedge |y| = o(\sqrt{n})$,

$$p_A^n(x, y) = \frac{(\sigma^2 a(x) - x)|y| + (\sigma^2 a(-y) + y)x}{\sigma^2 n} p^n(y - x) \{1 + o(1)\}. \quad (6.8)$$

If $E[|X|^3; X < 0] < \infty$, then (6.6) holds in view of (1.16). If $E[|X|^3; X < 0] = \infty$, by (2.17) of Lemma 2.1 the condition (6.6) may be expressed explicitly in terms of p and it in particular follows that if $F(x)$ is regularly varying at $-\infty$ with an exponent α less than -2 , then $\sigma^2 a(x) - x$ varies regularly with exponent $(\alpha + 3) \vee 0$, which is less than 1, hence (6.6) holds; in the critical case $\alpha = -2$ (with $\sum F(x)|x| < \infty$) $(\sigma^2 a(x) - x)/x$ becomes slowly varying so that (6.6) is violated.

By virtue of Proposition 5 Theorem 3 follows if we show

Proposition 6. *Suppose (6.6) to hold true. Let $y < 0 < x$ and suppose the walk is not left-continuous (so that $p_{\{0\}}^n(x, y)$ is not identically zero). Then for any $M > 1$, uniformly for $x, |y| \leq M\sqrt{n}$, as $x \wedge |y| \wedge n \rightarrow \infty$ under $(x \wedge |y|) = o(\sqrt{n})$*

$$p_{\{0\}}^n(x, y) = \frac{(\sigma^2 a(x) - x)|y| + (\sigma^2 a(-y) + y)x}{\sigma^2 n} p^n(y - x) \{1 + o(1)\}. \quad (6.9)$$

Proof. The proof parallels that given for the same formula but under the additional condition $E[|X|^3; X < 0] < \infty$ in [12]. For simplicity we suppose $\nu = 1$, i.e., the walk is aperiodic. Recall that $h_x(n, z)$ is the space-time hitting probability of $(-\infty, 0]$. Making decomposition

$$p_{\{0\}}^n(x, y) = \sum_{k=1}^n \sum_{z < 0} h_x(k, z) p_{\{0\}}^{n-k}(z, y), \quad (6.10)$$

we break the double sum into three parts by partitioning the range of the outer summation as follows

$$1 \leq k < \varepsilon n; \quad \varepsilon n \leq k \leq (1 - \varepsilon)n; \quad (1 - \varepsilon)n < k \leq n$$

and call the corresponding sums *I*, *II* and *III*, respectively. Here ε is a positive constant less than $1/4$ that will be chosen small.

We call

$$\lambda(x) := a(x) - \frac{x}{\sigma^2} \quad (6.11)$$

and suppose $\lambda(x) \rightarrow \infty$ as $x \rightarrow \infty$, or equivalently $E[|X|^3; X < 0] = \infty$; otherwise (6.9) is already verified in Theorem 2. This supposition permits using (2.18), which states

$$\sum_{z=-\infty}^{-1} H_{(-\infty, 0]}^{+\infty}(z)(|z| \wedge |y|) \sim \frac{\sigma^2}{2} \lambda(-y) \quad (y \rightarrow -\infty). \quad (6.12)$$

Let $x \wedge |y| = o(\sqrt{n})$. By duality one may suppose that $y = o(\sqrt{n})$. Then the hypothesis (6.6) says

$$|y| \lambda(\sqrt{n}) = \sqrt{n} \lambda(-y) \times o(1). \quad (6.13)$$

Proposition 1 implies

$$p_{\{0\}}^{n-k}(z, y) \leq \frac{C(|z| \wedge \sqrt{n})|y|}{n^{3/2}} \quad (k \leq \varepsilon n, z \leq -1),$$

whereas using Lemma 4.1 [(4.9, 4.10)] as well as (6.12) one deduces

$$\sum_{k \geq \varepsilon n} \sum_{z < 0} h_x(k, z)(|z| \wedge \sqrt{n}) \leq M_\varepsilon x \frac{\lambda(\sqrt{n})}{\sqrt{n}}. \quad (6.14)$$

By virtue of (6.13) we therefore obtain

$$II \leq M_\varepsilon \frac{x|y|}{n^{3/2}} \cdot \frac{\lambda(\sqrt{n})}{\sqrt{n}} = \frac{\lambda(-y)x}{n^{3/2}} \times o_\varepsilon(1)$$

Here (and below) M_ε designates a constant that may depend on ε but not on the other variables and $o_\varepsilon(1) \rightarrow 0$ as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ in this order uniformly in x, y (subject to the constraints in the proposition).

Similarly, on using Theorem A

$$\begin{aligned} I &= \sum_{1 \leq k < \varepsilon n} \sum_{-\sqrt{n}/\varepsilon \leq z < 0} h_x(k, z) \cdot \frac{\sigma^4 a(z) a(-y) + zy}{\sigma^2(n-k)} \mathbf{g}_{n-k}(y) (1 + o_\varepsilon(1)) \\ &\quad + \sum_{z < -\sqrt{n}/\varepsilon} H_{(-\infty, 0]}^x(z) \times O\left(\frac{y}{n}\right). \end{aligned} \quad (6.15)$$

To find an upper bound of the last sum we use the identity (2.15), which may be written as

$$\sum_{z < 0} H_{(-\infty, 0]}^x(z) \lambda(z) = \lambda(x). \quad (6.16)$$

By Markov's inequality and $\lambda(z) > -z/\sigma^2$ this entails

$$\sum_{z < -\sqrt{n}/\varepsilon} H_{(-\infty, 0]}^x(z) \leq \frac{\varepsilon}{\sqrt{n}} \sum_{z < -\sqrt{n}/\varepsilon} |z| H_{(-\infty, 0]}^x(z) \leq \frac{\varepsilon \sigma^2 \lambda(x)}{\sqrt{n}}. \quad (6.17)$$

For the evaluation of the double sum in (6.15) we may replace $(n-k)^{-1} \mathbf{g}_{n-k}(y)$ by $n^{-1} \mathbf{g}_n(y) (1 + O(\varepsilon))$. Since y is supposed to go to $-\infty$, we may also replace $\sigma^4 a(z) a(-y) + zy = \sigma^4 \lambda(z) a(-y) + \sigma^2 z \lambda(-y)$ by $\sigma^2 \lambda(z) |y|$ and in view of (6.14) we may extend the range of the outer summation to the whole half line $k \geq 1$. Using (6.13), (6.15), (6.16) and (6.17) we then deduce that

$$I = \frac{\lambda(x) |y| \mathbf{g}_n(y)}{n} [1 + O(\varepsilon) + o_\varepsilon(1)].$$

As for III observe that

$$\sum_{(1-\varepsilon)n < k \leq n} p_{\{0\}}^{n-k}(z, y) = g(z, y) - r_n(z, y) \quad (z < 0) \quad (6.18)$$

with $0 \leq r_n(z, y) := \sum_{j \geq \varepsilon n} p_{\{0\}}^j(z, y) \leq C(|z| \wedge \sqrt{n}) |y| / \sqrt{\varepsilon n}$, as deduced from Proposition 1 for $|z| < \sqrt{n}$ and by the bound $g(z, y) \leq O(y)$ for $|z| \geq \sqrt{n}$; hence by (6.14) and (6.13)

$$\sum_{k \geq (1-\varepsilon)n} \sum_{z < 0} h_x(k, z) r_n(z, y) \leq C \frac{x|y|}{n^{3/2}} \cdot \frac{\lambda(\sqrt{n})}{\sqrt{\varepsilon n}} = \frac{x \lambda(-y)}{n^{3/2}} \times o_\varepsilon(1).$$

By (6.18) it remains to evaluate the sum $\sum_{z < 0} h_x(k, z) g(z, y)$ uniformly over $(1-\varepsilon)n < k \leq n$. To this end we apply Lemma 4.1 again. In formula (4.9) of it the error term $\alpha_n(x, y)$ satisfies

$$\sum_{z < y} \left[\sup_{(n/2 < k \leq n} \alpha_k(x, z) \right] g(z, y) \leq c_M |y| \frac{\lambda(-\varepsilon n)}{\varepsilon n} + \frac{\lambda(-y)}{y} \times o_\varepsilon(1)$$

according to (4.11). Now using $\mathbf{g}_k(x) = \mathbf{g}_n(x)(1 + O(\varepsilon))$ ($(1 - \varepsilon)n < k \leq n$) as well as $y = o(\sqrt{n})$ we apply (6.12) and (4.9) to find that

$$III = \frac{f^+(x)\mathbf{g}_n(x)}{\sigma^2 n} \sum_{z < 0} H_{(-\infty, 0]}^{+\infty}(z)g(z, y)[1 + o_\varepsilon(1) + O(\varepsilon)],$$

hence by the identity (2.14)

$$III = \frac{x\lambda(-y)\mathbf{g}_n(x)}{n}[1 + o_\varepsilon(1) + O(\varepsilon)].$$

Adding these contributions yields the desired formula, since ε can be made arbitrarily small. \square

In view of (2.16) $\lambda(x)/x$ is asymptotically decreasing (as $x \rightarrow \infty$) in the sense that $\lambda(x)/x \sim \mu(x)$ with a decreasing μ , as is noted previously. Keeping this in mind one examines the proof above and deduces the following upper bound without assuming (6.6).

Proposition 7. *There exists a constant C such that*

$$p_{\{0\}}^n(x, y) \leq C \frac{x\lambda(-y) + |y|\lambda(x)}{n^{3/2}} \quad (-M\sqrt{n} < y < 0 < x < M\sqrt{n}). \quad (6.19)$$

REMARK 6. One can show that $p_{\{0\}}^n(x, y)$ may be of smaller order of the right side of (6.19) for suitably chosen x, y with $x \vee |y| = o(\sqrt{n})$ if e.g. $P[X < -z] \sim 1/z^2(\log z)^2$ ($z \rightarrow +\infty$). Thus formula (1.4) is not generally true for $|x| \vee |y| < M\sqrt{n}$, $xy < 0$.

The next result provides a lower bound without assuming (6.6).

Proposition 8. *Given $M \geq 1$, for $-M\sqrt{n} < y < 0 < x < M\sqrt{n}$ satisfying (1.3),*

$$p_A^n(x, y) \geq c \left(\sum_{w=2}^{x \wedge |y|} p(-w)w^3 \right) \frac{x + |y|}{\sigma^4 n} p^n(y - x),$$

where c is a positive constant depending only on M/σ .

If $F(y)$ is regularly varying as $y \rightarrow -\infty$ with exponent $\alpha \in [-3, -2]$ and $E[|X|^3; X < 0] = \infty$, then in the non-critical case $\alpha \neq -2$ we have $\sum_{w=2}^x p(-w)w^3 \asymp \lambda(x)$ and hence owing to (6.6)

$$(x + |y|) \sum_{w=2}^{x \wedge |y|} p(-w)w^3 \asymp x\lambda(-y) + |y|\lambda(x) \quad \text{as } x \wedge (-y) \rightarrow \infty$$

so that the above lower bound is exact, whereas in the critical case $\alpha = -2$ when $w \int_w^\infty F(-t)dt$ is slowly varying at infinity $\sum_{w=2}^{x \wedge (-y)} p(-w)w^3 = o(\lambda(x))$ ($x \rightarrow \infty$)—as is inferred by using (2.17)—so that the lower bound is consistent to what is mentioned in Remark 6.

Proof of Proposition 8. In view of Proposition 5 as well as Theorems 2 we may suppose $A = \{0\}$. We also suppose $\nu = 1$ for simplicity and $x \leq |y|$ for reason of duality. Let c_1, c'_1, c_2

etc. denote positive constants depending only on M/σ . Substitution from Theorem C and Theorem 1 into (6.10) shows that for a constant $\delta > 0$,

$$p_{\{0\}}^n(x, y) \geq c_1 \sum_{\delta x^2 \leq k \leq n/2} \sum_{1 \leq w \leq \sqrt{k/\delta}} \sum_{z=-x}^{-1} \frac{xw}{\sigma^2 k} p^k(w-x) p(z-w) \frac{zy}{\sigma^2(n-k)} p^{n-k}(y-z).$$

We take

$$\delta = (4M^2)^{-1}$$

so that $n/2(x-y)x > \delta$ hence $\delta x^2 = nx/2(x-y) \leq n/4$ due to $x \leq |y|$. For k, w, z taken from the range of summation above, in view of the local limit theorem

$$\begin{aligned} p^k(w-x) p^{n-k}(y-z) &\geq c_2 \mathbf{g}_k(w-x) \mathbf{g}_{n-k}(y-z) \\ &\geq c_2 \mathbf{g}_k(x) \mathbf{g}_{n-k}(y). \end{aligned}$$

If $0 < x < M\sqrt{n}$ and $k \geq \delta x^2$, then $x \leq \sqrt{k/\delta}$ so that $\mathbf{g}_k(x) \leq c'_2 e^{-1/2\delta\sigma^2} k^{-1/2}/\sigma$; and also $\mathbf{g}_{n-k}(y) \leq \mathbf{g}_{n/2}(y) \leq c''_2 p^n(y-x)$ ($k \leq n/2$). Hence, putting

$$m(x) = \sum_{w=1}^x \sum_{z=1}^x p(-z-w) wz$$

we have

$$p_{\{0\}}^n(x, y) \geq c_3 \frac{m(x) p^n(y-x) x |y|}{\sigma^4 n} \sum_{\delta x^2 \leq k \leq n/2} \frac{1}{\sigma k^{3/2}}.$$

Since $\delta x^2 \leq n/4$, the last sum is bounded below by $(2 - \sqrt{2})/x\sigma\sqrt{\delta}$. Finally an easy computation yields that $m(x) \geq \sum_{w=1}^x \sum_{z=1}^{x-w} p(-z-w) wz \sim \frac{1}{6} \sum_{w=2}^x p(-w) w^3$ ($x \rightarrow \infty$), which concludes the proof. \square

Corollary 7. *For each $M \geq 1$, it holds under the constraint $-M\sqrt{n} < y < 0 < x < M\sqrt{n}$ that*

$$\begin{aligned} &P_x[S_{\sigma(-\infty, 0]} < -\eta \mid \sigma_{\{0\}} > n, S_n = y] \\ &\longrightarrow \begin{cases} 0 & \text{as } \eta \rightarrow \infty \quad \text{uniformly for } x, y \quad \text{if } E[|X|^3; X < 0] < \infty, \\ 1 & \text{as } x \wedge (-y) \rightarrow \infty \quad \text{for each } \eta > 0 \quad \text{if } E[|X|^3; X < 0] = \infty. \end{cases} \end{aligned}$$

Proof. If $E[|X|; X < 0] < \infty$, then by $H_{(-\infty, 0]}^x(z) \leq CH_{(-\infty, 0]}^\infty(z)$ it follows that for any $w < 0$

$$\sum_{z < -\eta} H_{(-\infty, -\eta]}^x(z) (|z| \wedge |w|) \leq C \sum_{z < -\eta} H_{(-\infty, 0]}^{+\infty}(z) |z| \rightarrow 0 \quad (\eta \rightarrow \infty),$$

and following the proof of Proposition 6 one can readily show the first relation. The second one follows from Proposition 8 since the contribution to the sum (6.10) from $-\eta \leq z < 0$ is $O(\eta(x \vee |y|)/n^{3/2})$ as is easily verified (use $\sum_{k \geq n/2} p_{\{0\}}^{n-k}(z, y) \leq g_{\{0\}}(z, y) \leq C\eta$ for $z \leq -\eta$), hence negligible relative to the lower bound given by Proposition 8. \square

7 Appendices

A. A CONSEQUENCE OF DUALITY.

In (2.11) we have brought in Green's function

$$g(x, y) = a(x) + a(-y) - a(x - y) \quad (7.1)$$

on the space $\mathbb{Z} \setminus \{0\}$. By means of g the identity (3.2) may be written as

$$\begin{aligned} g_A(x, y) &= a^\dagger(x - \xi_0) - a(x - y) + E_x[a(S_{\sigma(A)} - y)] - E_x[a(S_{\sigma(A)} - \xi_0)] \\ &= \delta_{x, \xi_0} + g(x - \xi_0, y - \xi_0) - E_x[g(S_{\sigma(A)} - \xi_0, y - \xi_0)] \end{aligned} \quad (7.2)$$

(for any $\xi_0 \in A$). It is noted that for all x, y

$$g_A(x, y) = g(x, y) + O(1).$$

We consider the dual walk, denoted by \widehat{S}_n , that is a random walk with transition probability

$$P[\widehat{S}_n = y \mid \widehat{S}_0 = x] = p^n(x - y).$$

The objects associated with this walk are denoted by $\widehat{P}_x, \widehat{E}_x, \widehat{p}_A^n$, etc. Then for $x, y \notin A$,

$$\widehat{p}_A^n(y, x) = p_A^n(x, y) = p_{-A}^n(-y, -x), \quad (7.3)$$

the law of (\widehat{S}_n) with $\widehat{S}_0 = y$ being the same as that of the walk $(-S_n)$:

$$\widehat{p}_A^n(y, x) = P[-S_1 \notin A, \dots, -S_n \notin A, -S_n = x \mid -S_0 = y] = p_{-A}^n(-y, -x).$$

Hence $\widehat{g}_A(y, x) = g_A(x, y) = g_{-A}(-y, -x)$ ($x, y \notin A$). It is noted that the second equality in (7.3) follows directly from the invariance of the law of the walk under a reversal of the order of the increments X_1, \dots, X_n .

We prove the following result of which the first relation entails (1.18).

Lemma 7.1. $\lim_{x \rightarrow \infty} [g_A^-(x) - g_{-A}^-(x)] = 0$ and $\lim_{y \rightarrow -\infty} [g_A^+(y) - g_{-A}^+(y)] = 0$.

Proof. It suffices to show the first relation, the second being its dual. We may suppose $0 \in A$. Using the identity $g_{-A}(x, y) = g_A(-y, -x)$, (3.1) and (3.2) in turn we see that

$$\begin{aligned} g_A(x, y) - g_{-A}(x, y) &= u_A(x) - u_A(-y) + E_x[a(S_{\sigma(A)} - y)] - E_{-y}[a(S_{\sigma(A)} + x)] \\ &= a(x) - E_x[a(S_{\sigma(A)})] - a(-y) + E_{-y}[a(S_{\sigma(A)})] \\ &\quad + E_x[a(S_{\sigma(A)} - y)] - E_{-y}[a(S_{\sigma(A)} + x)], \end{aligned}$$

Letting $y \rightarrow -\infty$ we obtain

$$g_A^-(x) - g_{-A}^-(x) = a(x) - E_x[a(S_{\sigma(A)})] + \sum_{\xi} H_A^{+\infty}[a(\xi) + \xi/\sigma^2 - a(\xi + x)],$$

which tends to zero as $x \rightarrow +\infty$ as desired. □

B. SOME INEQUALITIES CONCERNING $a(x) - x/\sigma^2$.

Let $\lambda(x)$ be the function defined by (6.11) and $\widehat{\lambda}(x)$ its dual:

$$\lambda(x) = a(x) - \frac{x}{\sigma^2} \quad \text{and} \quad \widehat{\lambda}(x) = a(-x) - \frac{x}{\sigma^2}.$$

Here we collect several formulae satisfied by $\lambda(x)$ or $\widehat{\lambda}(x)$, all of which rest on the known results presented in Section 2; they are trivial when $a(x) - |x|/\sigma^2$ converges to a finite number as $x \rightarrow \infty$ or $x \rightarrow -\infty$ according to the situation. It is noted that

$$\lambda(x+y) - \lambda(y) = a(x+y) - a(y) - x/\sigma^2$$

and

$$\widehat{\lambda}(x+y) - \widehat{\lambda}(y) = a(-x-y) - a(-y) - x/\sigma^2.$$

(i) For $x > 0$ and $y \geq 0$,

$$-\lambda(x) \times o(1) \leq \lambda(x+y) - \lambda(y) \leq \lambda(x) \quad (7.4)$$

and

$$-\widehat{\lambda}(x) \times o(1) \leq \widehat{\lambda}(x+y) - \widehat{\lambda}(y) \leq \widehat{\lambda}(x), \quad (7.5)$$

where $o(1)$ is bounded and, as $y \rightarrow \infty$, tends to zero uniformly in x .

The relation (7.5) is a dual of (7.4). For the proof of (7.4) we may suppose that the walk is not left continuous so that $\lambda(x) > 0$ for $x > 0$. The second inequality of (7.4) is the same as $g(x, -y) \geq 0$. Using identity (2.15) as well as $g(x, -y) = \sum H_{(-\infty, 0]}^x(z)g(z, -y)$ we observe

$$\begin{aligned} -\lambda(x+y) + \lambda(y) &= g(x, -y) - [a(x) - x/\sigma^2] \\ &= \sum H_{(-\infty, 0]}^x(z) \left(g(z, -y) - [a(z) - z/\sigma^2] \right) \\ &= \sum H_{(-\infty, 0]}^x(z) [a(y) - a(y+z) + z/\sigma^2] \\ &\leq \sum H_{(-\infty, 0]}^x(z) [a(-z) + z/\sigma^2]. \end{aligned} \quad (7.6)$$

The left-most member tends to zero as $y \rightarrow \infty$ for each x while the right most is positive because of the non-left-continuity assumption. Hence the first inequality of (7.4) holds true for each x . To see the uniformity in x observe that $a(-z) + z/\sigma^2 = o(z)$ as $z \rightarrow -\infty$ whereas $a(z) - z/\sigma^2 \geq |z|/\sigma^2$, then you may apply (2.15) again, in the case when $\sum H_{(-\infty, 0]}^\infty(z)|z| = \infty$. In the other case we use the expression next to the last in (7.6) and apply the dominated convergence (or use the fact mentioned around (1.16) from the outset).

(ii) For $0 < x \leq y$,

$$-\widehat{\lambda}(x) \leq \widehat{\lambda}(y-x) - \widehat{\lambda}(y) \leq \widehat{\lambda}(x) \times o(1), \quad (7.7)$$

where $o(1)$ is bounded and, as $y-x \rightarrow \infty$, tends to zero uniformly in x .

On putting $y' = y-x$ and writing $\widehat{\lambda}(y-x) - \widehat{\lambda}(y) = \widehat{\lambda}(y') - \widehat{\lambda}(y'+x)$, there follows from (7.5) the formula (7.7) with the second inequality restricted to the case when $y' \rightarrow \infty$.

(iii) If $|x| \leq y$,

$$|\widehat{\lambda}(y-x) - \widehat{\lambda}(y)| \leq C\widehat{\lambda}(|x|). \quad (7.8)$$

This follows from (7.5) if $x < 0$ and from (7.7) if $x \geq 0$.

C. COMPARISON BETWEEN p_A^n AND $p_{\{0\}}^n$.

What is asserted in Remark 2 for x, y subject to $|x| \wedge |y| = O(1)$, not immediate from Theorem 1, is shown in this appendix. As in Remark 2 let $0 \in A$ and $\sharp A \geq 2$ (one may suppose $\sharp A = 2$ if he wishes) and suppose the walk is aperiodic and $|x| \vee |y| = O(\sqrt{n})$.

Suppose that $y \notin A$ and $p_{\{0\}}^n(x, y) > 0$ for some $n \geq 1$ (hence for all sufficiently large n). Then for each (admissible) x and y , the ratio $p_A^n(x, y)/p_{\{0\}}^n(x, y)$ converges to

$$\frac{g_A^+(x)g_{-A}^-(-y) + g_A^-(x)g_{-A}^+(-y)}{g_{\{0\}}^+(x)g_{\{0\}}^-(-y) + g_{\{0\}}^-(x)g_{\{0\}}^+(-y)} \quad (7.9)$$

as $n \rightarrow \infty$, and this ratio converges to

$$\frac{g_{-A}^-(-y)}{g_{\{0\}}^-(-y)} = 1 - \frac{E_{-y}[\sigma^2 a(S_{\sigma(-A)}) - S_{\sigma(-A)}]}{\sigma^2 a(-y) + y} \quad (7.10)$$

as $x \rightarrow +\infty$ for each y (with $\sigma^2 a(-y) + y \neq 0$) and

$$\frac{g_A^-(x)}{g_{\{0\}}^-(x)} = 1 - \frac{E_x[\sigma^2 a(S_{\sigma(A)}) + S_{\sigma(A)}]}{\sigma^2 a^\dagger(x) + x} \quad (7.11)$$

as $y \rightarrow -\infty$ for each x (with $\sigma^2 a^\dagger(x) + x \neq 0$) and similarly for the cases $x \rightarrow -\infty$ and $y \rightarrow +\infty$. What is presently required is to prove that the ratios on the left sides in (7.9) and in (7.10) are less than unity unless $p_A^n(x, y) = p_{\{0\}}^n(x, y)$ for all n . It suffices to show that those in (7.10) and (7.11) are less than unity, this entailing that the ratio (7.9) is also less than unity since $g_A^\pm(x) \leq g_{\{0\}}^\pm(x)$.

For reason of duality we suppose $x \geq 0$ and $|y| < M$ for an arbitrarily chosen $M > 1$. We have $p_A^n(x, y) < p_{\{0\}}^n(x, y)$ for some $n \geq 1$ if and only if the walk from x to y can pass through A before visiting 0 with a positive probability, provided $xy \neq 0$. For $x > 0$, this is the case if and only if the following condition holds

$$(*) \left\{ \begin{array}{ll} y > 0 \text{ and } A \cap [1, \infty) \neq \emptyset & \text{if } S \text{ is left-continuous,} \\ \text{either } y > 0 \text{ and } A \cap [1, \infty) \neq \emptyset \\ \text{or } y < 0 \text{ and } S \text{ is not left-continuous} & \end{array} \right\} \quad \text{if } S \text{ is right-continuous.}$$

The case $x = 0$ may be reduced to the case $x \neq 0$ and obviously $p_A^n(0, y) < p_{\{0\}}^n(0, y)$ ($y \neq 0$) for some $n \geq 1$ under (*). Now, using relation (2.1), one can easily check that the required inequality is true.

Acknowledgments. I wish to thank anonymous referees for their helpful comments that motivate the author to improve the paper in various aspects in revision.

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